Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

[Part marks]

1. In the Schrödinger picture, the state vector $|\psi(t)\rangle$ of a physical system with a Hamiltonian H_0 evolves according to

$$|\psi(t)\rangle = T(t,t_0)\,|\psi(t_0)\rangle$$

where $T(t,t_0)$ is a unitary evolution operator and t_0 an arbitrary time. Show that $T(t,t_0)$ satisfies the differential equation

$$i\hbar\frac{\partial T(t,t_0)}{\partial t} = H_0 T(t,t_0)\,.$$

Show that in the case that H_0 is independent of time, then

$$T(t,t_0) = \mathrm{e}^{-iH_0(t-t_0)/\hbar}.$$

Discuss briefly the difference in treatment of the time evolution of a system in the Schrödinger and Heisenberg pictures. How are the state vectors in the two pictures related to each other? **[5]**

An operator $A_H(t)$ of the Heisenberg picture is obtained from that $A_S(t)$ of the Schrödinger picture by the transform

$$A_H(t) = T^{\dagger}(t,t_0)\,A_S(t)\,T(t,t_0)\,.$$

Show that $A_H(t)$ obeys the Heisenberg equation of motion,

$$\frac{\mathrm{d}A_H(t)}{\mathrm{d}t} = \frac{1}{i\hbar}\left[A_H(t)\,, H_H(t)\right] + \left(\frac{\partial A}{\partial t}\right)_H$$

where H_H is the Hamiltonian in the Heisenberg picture. **[6]**

The time-independent Hamiltonian for the interaction of a spin-$\frac{1}{2}$ particle in a uniform magnetic field of induction $\mathbf{B}$ along the z-axis is

$$H = \gamma B S_z, \qquad \gamma \text{ a constant.}$$

Establish the equations of motion for the time-dependent spin operators $S_x(t)$, $S_y(t)$ and $S_z(t)$ in the Heisenberg picture. **[5]**

Solve the equations of motion to obtain these operators as explicit functions of time. **[4]**

2. A solution of the time-dependent Schrödinger equation can be written in the form

$$\psi(\mathbf{r},t) = \sum_n c_n(t)\,\phi_n(\mathbf{r})\,\mathrm{e}^{-iE_n t/\hbar},$$

where $\phi_n(\mathbf{r})$ are eigenfunctions with eigenvalues E_n of a time-independent Hamiltonian H_0. What physical interpretation can be placed on the coefficients $c_n(t)$?

The Hamiltonian describing a particular system is

$$H(\mathbf{r},t) = H_0(\mathbf{r}) + \lambda H'(\mathbf{r},t),$$

where λ is a small, real parameter. If at time t_0, the system is in a definite eigenstate $\phi_i(\mathbf{r})$ with energy E_i, show that at a later time t the transition amplitude for excitation of a state $\phi_k(\mathbf{r})$ of energy E_k ($\neq E_i$), to lowest order in λ, is given by the expression

$$\frac{1}{i\hbar}\int_{t_0}^{t}\langle\phi_k(\mathbf{r})\,|\lambda H'(\mathbf{r},t')\,|\phi_i(\mathbf{r})\rangle\,\mathrm{e}^{i\omega_{ki}t'}\,\mathrm{d}t',$$

with $\hbar\omega_{ki} = (E_k - E_i)$ **[10]**

A particle of mass m is initially, for $t < 0$, in its ground state in an infinite one-dimensional well

$$V(x) = \begin{cases} 0 & |x| < a \\ \infty & |x| \geq a \end{cases}.$$

It is acted on for $t \geq 0$ by a time-dependent perturbation

$$H'(x,t) = \lambda x\,\mathrm{e}^{-\gamma t}, \qquad \gamma > 0.$$

To which states can a first-order transition occur? **[3]**

Calculate the probability of the transition, as $t \to \infty$, that has the largest probability. **[7]**

{The orthonormal eigenfunctions for a particle in the infinite one-dimensional potential well defined above are

$$\psi_n(x) = \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi x}{2a}\right) \qquad n = 1, 3, 5\ldots,$$
$$\psi_n(x) = \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi x}{2a}\right) \qquad n = 2, 4, 6\ldots,$$

with corresponding energies $E_n = \dfrac{n^2\pi^2\hbar^2}{8ma^2}$.

Note:

$2\sin A\cos B = \sin(A-B) + \sin(A+B)$; and $2\cos A\cos B = \cos(A-B) + \cos(A+B)$;

$\int x\sin bx\,\mathrm{d}x = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx + C$; and $\int x\cos bx\,\mathrm{d}x = \frac{1}{b^2}\cos bx + \frac{x}{b}\sin bx + C$. }

3. $\mathbf{J}$ is an angular momentum whose Cartesian components J_x, J_y and J_z are Hermitian operators obeying the commutation relations

$$[J_x, J_y] = i\hbar J_z; \quad [J_y, J_z] = i\hbar J_x; \quad [J_z, J_x] = i\hbar J_y.$$

Raising and lowering operators J_+ and J_- are defined by

$$J_+ = J_x + iJ_y; \quad \text{and} \quad J_- = J_x - iJ_y.$$

Show that

$$[J_z, J_\pm] = \pm\hbar J_\pm,$$

$$J_\pm J_\mp = \mathbf{J}^2 - J_z^2 \pm \hbar J_z.$$ [3]

If $|j, m\rangle$ are the simultaneous orthonormal eigenstates of $\mathbf{J}^2$ and J_z, with

$$\mathbf{J}^2|j, m\rangle = j\,(j+1)\,\hbar^2|j, m\rangle \quad \text{and} \quad J_z|j, m\rangle = m\hbar|j, m\rangle,$$

show that

$$J_-|j, m\rangle = \sqrt{j\,(j+1) - m\,(m-1)}\,\hbar\,|j, m-1\rangle.$$ [5]

In atomic hydrogen, the spin-orbit interaction couples the orbital angular momentum $\mathbf{L}$ and spin $\mathbf{S}$ of the electron to form a resultant angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Denote the uncoupled eigenstates of $\mathbf{L}^2$, $\mathbf{S}^2$, L_z and S_z by $|\ell, \frac{1}{2}, m_\ell, m_s\rangle$ and the coupled eigenstates of $\mathbf{L}^2$, $\mathbf{S}^2$, $\mathbf{J}^2$ and J_z by $|J, M\rangle$. If $\ell = 1$, show that

$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1, \tfrac{1}{2}, 0, \tfrac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|1, \tfrac{1}{2}, 1, -\tfrac{1}{2}\right\rangle$$

and explicitly obtain all the normalized eigenstates for the $j = 3/2$ state in terms of the uncoupled states. Outline a procedure for generating the coupled eigenstates for $j = 1/2$ but do **not** explicitly calculate them. [8]

A weak magnetic field of induction $\mathbf{B}$ along the z-axis introduces an additional term

$$H_Z = \frac{\mu_B B}{\hbar}\,(J_z + S_z)$$

into the Hamiltonian, where μ_B is the Bohr magneton. If the states $|J, M\rangle$, which diagonalize the calculation of the dominant spin-orbit perturbation, are used in a first-order perturbation calculation, H_z makes a contribution,

$$E_Z^{(1)} = \frac{\mu_B B}{\hbar}\,\langle J, M_J\,|J_z + S_z|\,J, M_J\rangle\,,$$

to the energy levels. Show that for the $\mathrm{P}_{3/2}$ state, this contribution may be summarised by the equation

$$E_Z^{(1)} = \frac{4}{3}\mu_B B M_J.$$ [4]

4. The time-independent Schrödinger equation for a particle of mass m and energy E moving along the x-axis is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+V(x)\,\psi(x)=E\psi(x)\,,$$

where $\psi(x)$ is the wave function and $V(x)$ is the potential. If $\psi(x)$ is written in the form $\psi(x)=\mathrm{e}^{iu(x)}$, show that $u(x)$ satisfies the differential equation

$$\left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^2=k^2(x)+i\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2},$$

where $k^2(x)=\frac{2m}{\hbar^2}(E-V(x))$, for $E>V(x)$. **[2]**

If $V(x)$ varies slowly with x, show that a first approximation for $u(x)$ is

$$u_0(x)=\pm\int^x k(x)\,\mathrm{d}x+C_0,$$

and that the next iteration is

$$u_1(x)=\pm\int^x\sqrt{k^2(x)\pm ik'(x)}\,\mathrm{d}x+C_1.$$

Hence show that an approximation for the wave function is

$$\psi(x)=\frac{1}{\sqrt{k(x)}}\exp\left\{\pm i\int^x k(x)\,\mathrm{d}x\right\},$$

provided $|k'(x)|\ll|k^2(x)|$. **[5]**

A particle moves in a well-shaped potential with the two classical turning points at $x=b$ and $x=a$ as shown in figure 1.

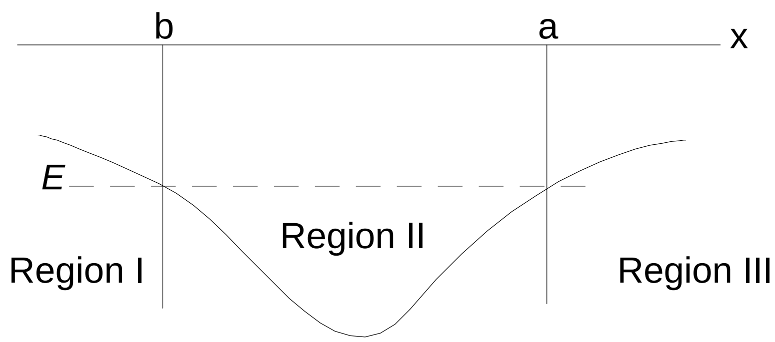

Figure 1. A one-dimensional potential well.

QUESTION CONTINUED

The connection formulae (C1 - C4) below link the WKB wave functions between the regions with real wave number, (II), and imaginary wave number, (I, III). Show how these wave functions can be used to obtain the condition

$$\int_b^a \left[\frac{2m}{\hbar^2}\left(E_n - V(x)\right)\right]^{1/2} \mathrm{d}x = \left(n + \tfrac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots, \qquad \textbf{[6]}$$

from which an approximation for the bound state energies may be found.

A particle of mass m is confined by the potential

$$V(x) = \begin{cases} -V_0\left(1 - |x|/L\right) & |x| \le L, \qquad V_0 > 0, \\ 0 & |x| > L. \end{cases}$$

Obtain an approximation for the bound state energies. Hence find a relationship between V_0 and L for there to be at least one bound state. **[7]**

{The connection formulae:

The L.H.S. is valid for $x \ll a$, the R.H.S. is valid for $x \gg a$, i.e. sufficiently far away from the turning point $x = a$. .

$$\frac{2A}{\sqrt{k(x)}}\cos\left(\int_x^a k(x)\,\mathrm{d}x - \frac{\pi}{4}\right) \leftarrow \frac{A}{\sqrt{q(x)}}\exp\left\{-\int_a^x q(x)\,\mathrm{d}x\right\} \qquad \text{(C.1)}$$

$$\frac{B}{\sqrt{k(x)}}\sin\left(\int_x^a k(x)\,\mathrm{d}x - \frac{\pi}{4}\right) \rightarrow \frac{-B}{\sqrt{q(x)}}\exp\left\{\int_a^x q(x)\,\mathrm{d}x\right\} \qquad \text{(C.2)}$$

The L.H.S. is valid for $x \ll b$, the R.H.S. is valid for $x \gg b$, i.e. sufficiently far away from the turning point $x = b$.

$$\frac{A}{\sqrt{q(x)}}\exp\left\{-\int_x^b q(x)\,\mathrm{d}x\right\} \rightarrow \frac{2A}{\sqrt{k(x)}}\cos\left(\int_b^x k(x)\,\mathrm{d}x - \frac{\pi}{4}\right) \qquad \text{(C.3)}$$

$$\frac{-B}{\sqrt{q(x)}}\exp\left\{\int_x^b q(x)\,\mathrm{d}x\right\} \leftarrow \frac{B}{\sqrt{k(x)}}\sin\left(\int_b^x k(x)\,\mathrm{d}x - \frac{\pi}{4}\right) \qquad \text{(C.4)}$$

with $k(x) = \sqrt{\frac{2m}{\hbar^2}\left(E - V(x)\right)}$ for $E > V(x)$ and $q(x) = \sqrt{\frac{2m}{\hbar^2}\left(V(x) - E\right)}$ for $V(x) > E$.

Note also:

$\cos(A - B - \pi/4) = \sin A \cos(B - \pi/4) - \cos A \sin(B - \pi/4)$. }

5. If the wave function $\psi(r,\theta)$ describing the motion of a spinless particle of mass m and energy E in a central potential $V(r)$ is expanded as

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{u_\ell}{r}(r) P_\ell(\cos\theta),$$

where $P_\ell(\cos\theta)$ are Legendre polynomials, then $u_\ell(r)$ satisfies the reduced radial wave equation

$$\frac{\mathrm{d}^2 u_\ell(r)}{\mathrm{d}r^2} + \left\{k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r)\right\} u_\ell(r) = 0,$$

with $k^2 = 2mE/\hbar^2$. If $\mathbf{k}$ lies along the z-axis, show that the function

$$\psi(r,\theta) = \beta\mathbf{k}\cdot\boldsymbol{\nabla}\left(\frac{\mathrm{e}^{ikr}}{r}\right),$$

with β a complex quantity independent of r and θ, satisfies this differential equation outside the range of the potential for some value of ℓ. **[5]**

State the asymptotic form of the wave function representing the elastic scattering of a monochromatic beam of spinless particles by a finite-ranged central potential. Explain the physical interpretation of the terms in the wave function. **[3]**

Show that $\beta\mathbf{k}\cdot\boldsymbol{\nabla}\left(\frac{\mathrm{e}^{ikr}}{r}\right)$ represents an outgoing p-wave. Derive an expression for β in terms of the p-wave phase shift δ_1. What is the p-wave scattering length? **[6]**

Obtain the differential cross section. Show that the total cross section is $\frac{4\pi}{3}|\beta|^2 k^4$. **[6]**

{Note:
$P_1(x) = x$; and $\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$.

You may assume standard results from the partial wave analysis treatment of scattering of a monochromatic beam of particles from a central potential. }

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