Queen Mary
University of London

M. Sci. Examination by course unit 2010

MTH720U/MTHM033 Relativity and gravitation. SOLUTIONS

Duration: 3 hours

Date and time: xx xxx 2010, xxxxh

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 *Prove that the metric tensor is symmetric. Explain how this symmetry and dimensions of space-time pre-determine the total number of the Einstein Field Equations required for the description of space-time geometry.*

Solution 1 *[Seen similar]*

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ik}dx^i dx^k}{2} = \frac{g_{ik}dx^i dx^k + g_{ik}dx^k dx^i}{2}.$$

The following substitution in the second term

$$i \to k \;\; k \to i$$

gives

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ki}dx^i dx^k}{2} = \frac{g_{ik} + g_{ki}}{2} dx^i dx^k = \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{g_{ik} + g_{ki}}{2}.$$

[4]

Obviously that

$$\tilde{g}_{ik} = \tilde{g}_{ki}.$$

We can use $\tilde{g}_{ik}$ instead g_{ik} and then changing notations just drop $\tilde{}$. [2]

Space-time is four dimensional, i.e g_{ik} have 4×4 component. Due to the symmetry there only $4+3+2+1 = 10$ independent components. Hence, to describe geometry of 4-space time one needs 10 equations. [2]

Question 2 *Give the definition of the contravariant metric tensor g^{ik}. What manipulations with indices can be produced with the help of g_{ik} and g^{ik}? Show that in an arbitrary non-inertial frame*

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from a locally inertial frame of reference (local Galilean frame) to this non-inertial frame.

Solution 2 *[Seen similar]*

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta^l_i.$$

[1]

We can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta^l_i.$$

[1]

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k.$$

[1]

We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1).$$

[2]

Hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

[3]

Question 3 *Transformation from a local inertial (or local Galilean) frame of reference* $x^i_{(0)}$ *to some non-inertial frame* x^i *is given by the following transformation matrix:* $S^i_{(0)k} = \delta^i_k + f\delta^i_0\delta^0_k$, *where* $f = f(x^m_{(0)})$ *is a scalar field. Using the result of Question 1, show that the metric in the non-inertial frame of reference* x^i *has the following form:* $ds^2 = (1+f)^{-2}(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.

Solution 3 *[Seen similar]*

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{nm} = (\delta^i_n + f\delta^i_0\delta^0_n)(\delta^k_m + f\delta^k_0\delta^0_m)\eta^{nm} =$$

$$= [\delta^i_n\delta^k_m + f(\delta^i_n\delta^k_0\delta^0_m + \delta^k_m\delta^i_0\delta^0_n) + f^2\delta^i_0\delta^0_n\delta^k_0\delta^0_m]\eta^{nm} =$$

$$= \eta^{ik} + f(\eta^{i0}\delta^k_0 + \eta^{0k}\delta^i_0) + f^2\eta^{00}\delta^i_0\delta^k_0 = \eta^{ik} + 2f\delta^i_0\delta^k_0 + f^2\delta^i_0\delta^k_0 =$$

$$= \begin{pmatrix} (1+f)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[5]

Determinant $|g^{ik}| = -(1+f)^2$, hence g_{ik} which is reciprocal to g^{ik}, is presented by inverse matrix:

$$g_{ik} = \begin{pmatrix} (1+f)^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[2]

Finally

$$ds^2 = (1+f)^{-2}(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

[1]

Question 4 *Explain why in order to prove that some tensor is identically equal to zero it is enough to show that all components of this tensor are equal to zero in the local galilean frame of reference. Then, prove that the Christoffel symbols, Γ^i_{kl}, are symmetric with respect to their low indices.*

Solution 4. *[Seen similar]*

Transformation of tensors from the local galilean frame of reference to an arbitrary frame of reference is produced with the help of matrices S^i_k and $\tilde{S}^i_k$. It does not matter how many times these matrices appear in the transformation law, the resulting components of the tensor in a new frame of references are linear combinations of the components in the local galilean frame of reference, hence all new components are automatically equal to zero in an arbitrary frame of reference, if they are zero in the local galilean frame of reference. [2]

Let $A_i = \phi_{,i}$, where ϕ is a scalar, then

$$A_{i;k} - A_{k;i} = A_{i,k} - \Gamma^m_{ik} A_m - A_{k,i} + \Gamma^m_{ki} A_m = \phi_{,i,k} - \phi_{,k,i} + (\Gamma^m_{ki} - \Gamma^m_{ik}) A_m = (\Gamma^m_{ki} - \Gamma^m_{ik}) A_m.$$

[3]

LHS is a tensor. In a local galilean coordinates RHS=0, hence in the local galilean coordinates LHS=0. Thus LHS=0 in all coordinates, and finally taking into account that A_m is an arbitrary vector, we conclude that $\Gamma^m_{ki} - \Gamma^m_{ik} = 0$ in all coordinates, hence $\Gamma^m_{ki} = \Gamma^m_{ik}$. [3]

Question 5 *Prove that all covariant derivatives of the metric tensor are equal to zero, - i.e.,* $g_{ik;l} = 0$*. Then, using the symmetry of the Christoffel symbols proofed in question 4, show that the Christoffel symbols in terms of the metric tensor are* $\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$.

Solution 5. *[Seen similar]*

Let A_i is an arbitrary covariant vector. By the definition of D one can say that DA_i is also vector and its contravariant representation is

$$DA^i = g^{ik} DA_k.$$

[1]

On other hand

$$DA^i = D(g^{ik} A_k) = Dg^{ik} A_k + g^{ik} DA_k,$$

hence

$$g^{ik} DA_k = Dg^{ik} A_k + g^{ik} DA_k$$

which means that

$$Dg^{ik} A_k = 0$$

for arbitrary vector A_k, hence

$$Dg^{ik} = 0.$$

[2]

This means that

$$Dg_{ik} = g_{ik;l} dx^l = 0,$$

for arbitrary dx^l, which means that all $g_{ik;l} = 0$. [1]

We can apply covariant differentiation to g_{ik}:

$$g_{ik;l} = g_{ik,l} - \Gamma^m_{il} g_{mk} - \Gamma^m_{kl} g_{mi} = 0,$$

or after two cycling permutations of indices $i \to k$ $k \to l$ $l \to i$ we have

$$g_{ik,l} = \Gamma^m_{il} g_{mk} + \Gamma^m_{kl} g_{mi},$$

$$g_{kl,i} = \Gamma^m_{ki} g_{mk} + \Gamma^m_{li} g_{mk},$$

$$-g_{li,k} = -\Gamma^m_{lk} g_{ml} - \Gamma^m_{ik} g_{ml}.$$

[3]

Taking into account that $\Gamma^m_{ki} = \Gamma^m_{ik}$ and $g_{ik} = g_{ki}$we obtain by summation of LHSs and RHSs that

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right).$$

[1]

Question 6 *Prove that the determinant of the metric tensor, g, is negative in all frames of reference. Then, prove the following identity:*

$$2d\ln\sqrt{-g} = g^{ik}dg_{ik} = -g_{ik}dg^{ik}.$$

Solution 6 *[Seen similar]*

Taking into account that g_{ik} and g^{ik} are reciprocal, one obtains that $g \equiv det(g_{ik}) = 1/det(g^{ik})$. [1]

We know (see question 3) that

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{lm}.$$

Obviously, $det(\eta^{lm} = -1$, hence

$$det(g^{ik}) = det S^i_{(0)n} \times det S^k_{(0)m} \times det(\eta^{lm}) = -S^2,$$

where S is the determinant of the transformation matrix. One can see that $g = -S^{-2} < 0$ in all frames of reference. [2]

The determinant g depends on all components g_{ik}. Calculating g with the help, say the first raw, one can write

$$g = M^{1i} g_{1i},$$

where M^{1i} are minors of the components in the first row. Obviously M^{1i} do not contain g_{1i}. Hence

$$\frac{\partial g}{\partial g_{1i}} = M^{1i}.$$

This is true for any row in determinant, thus

$$\frac{\partial g}{\partial g_{ni}} = M^{ni}.$$

[2]

Taking into account that g^{ik} is inverse matrix of g_{ik}, one can write $g^{ik} = M^{ik}/g$, i.e. $M^{ik} = gg^{ik}$. Thus

$$dg = \frac{\partial g}{\partial g_{ik}} dg_{ik} = M^{ik} dg_{ik} = gg^{ik} dg_{ik},$$

hence

$$g^{ik} dg_{ik} = \frac{dg}{g} = d\ln|g| = d\ln(-g) = 2\ln\sqrt{-g}.$$

[2]

Then

$$g^{ik} dg_{ik} = d(g^{ik} g_{ik}) - g_{ik} dg^{ik} = d\delta^i_i - g_{ik} dg^{ik} = -g_{ik} dg^{ik}.$$

[1]

Question 7 *Consider a light ray (electromagnetic signal) propagating in a gravitational field. The four-dimensional wave vector for the electromagnetic signal is defined as $k^i = dx^i/d\lambda$, where λ is some parameter varying along the ray. The scalar function Ψ is called the eikonal and defined as $k_i = \Psi_{,i}$. Derive the Eikonal equation (i.e., the equation for Ψ) and explain how using this equation one can describe the propagation of electromagnetic signals in a given gravitational field.*

Solution 7. *[Seen similar]*

We know that along any light ray $ds^2 = g_ikdx^idx^k = 0$. [1]

Thus we have

$$g_{ik}k^ik^k = g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda} = \frac{g_{ik}dx^idx^k}{d\lambda^2} = \frac{ds^2}{d\lambda^2} = 0.$$

[2]

For an arbitrary covariant vector $k_i = g_{ik}k^k$ we can find such a scalar that $k_i = \Psi_{,i}$. [1]

Substituting this to the previous formula we obtain the following equation for Ψ:

$$g^{ik}S_{,i}S_{,k} = 0.$$

This equation is called the Eikonal equation. [2]

The Eikonal equation "works" in the following way:
(i) We solve this single equation for single scalar field $\Psi(x^m)$;
(ii) Taking partial derivatives we calculate covariant components of the four-dimensional wave vector $k_i = -\Psi_{,i}$;
(iii) With the help of g^{ik} we obtain contravariant components of the four-dimensional wave vector
(iv) Finally we calculate world lines of electromagnetic signals:

$$x^i(s) = \int k^i d\lambda.$$

[3]

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8 (a) *Prove the following identities:*

$$\Gamma^i_{ik} - (\ln\sqrt{-g})_{,k} = 0 \ \ \text{and} \ \ [\sqrt{-g}g^{ik})]_{,k} + \sqrt{-g}g^{kl}\Gamma^i_{kl} = 0.$$

[13]

Solution B1a *[Seen similar]*

$$\Gamma^i_{ik} = \frac{1}{2}g^{in}\left(g_{in,k} + g_{kn,i} - g_{ik,n}\right) = \frac{1}{2}g^{in}g_{in,k} + \frac{1}{2}g^{in}g_{kn,i} - \frac{1}{2}g^{in}g_{ik,n},$$

changing indices of summation in the last term, $i \to n$, $n \to i$, one obtains

$$\Gamma^i_{ik} = \frac{1}{2}g^{in}g_{in,k} + \frac{1}{2}g^{in}g_{kn,i} - \frac{1}{2}g^{ni}g_{nk,i} = \frac{1}{2}g^{in}g_{in,k} = (\ln\sqrt{-g})$$

(see question 6). Hence

$$\Gamma^i_{ik} - (\ln\sqrt{-g})_{,k} = 0.$$

[4]

$$g^{kl}\Gamma^i_{kl} = \frac{1}{2}g^{kl}g^{in}\left(g_{kn,l} + g_{ln,k} - g_{kl,n}\right) = \frac{1}{2}g^{kl}g^{in}g_{kn,l} + \frac{1}{2}g^{kl}g^{in}g_{ln,k} - \frac{1}{2}g^{kl}g^{in}g_{kl,n},$$

changing indices of summation in the second term, $k \to l$, $l \to k$, one obtains

$$g^{kl}\Gamma^i_{kl} = \frac{1}{2}g^{kl}g^{in}g_{kn,l} + \frac{1}{2}g^{lk}g^{in}g_{kn,l} - \frac{1}{2}g^{kl}g^{in}g_{kl,n} = g^{kl}g^{in}g_{kn,l} - \frac{1}{2}g^{in}\frac{dg}{g}$$

(see question 6). [4]

Then taking into account that

$$\frac{(\sqrt{-g})_{,n}}{\sqrt{-g}} = \frac{1}{2}\frac{-g_{,n}}{\sqrt{-g}\sqrt{-g}} = \frac{1}{2}\frac{-g_{,n}}{-g} = \frac{g_{,n}}{2g},$$

one obtains

$$g^{kl}\Gamma^i_{kl} = g^{in}\left(g^{kl}g_{kn,l} - \frac{(\sqrt{-g})_{,n}}{\sqrt{-g}}\right) = \frac{g^{in}}{\sqrt{-g}}\left(\sqrt{-g}g^{kl}g_{kn,l} - (\sqrt{-g})_{,n}\right) =$$

$$= \frac{g^{in}}{\sqrt{-g}}\left[\sqrt{-g}(g^{kl}g_{kn})_{,l} - g_{kn}g^{kl}_{,l} - (\sqrt{-g})_{,n}\right] = \frac{1}{\sqrt{-g}}\left[\sqrt{-g}(g^{in}\delta^k_n)_{,l} - g^{in}g_{kn}g^{kl}_{,l} - g^{in}(\sqrt{-g})_{,n}\right] =$$

$$= \frac{1}{\sqrt{-g}}\left[-\sqrt{-g}\delta^i_k g^{kl}_{,l} - g^{in}(\sqrt{-g})_{,n}\right] = \frac{1}{\sqrt{-g}}\left[-\sqrt{-g}g^{in}_{,n} - g^{in}(\sqrt{-g})_{,n}\right] = -\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{in}\right)_{,n},$$

hence

$$[\sqrt{-g}g^{ik})]_{,k} + \sqrt{-g}g^{kl}\Gamma^i_{kl} = 0.$$

[5]

(b) *Prove that the covariant divergence of an arbitrary contravariant vector can be written as*

$$A^i_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^i)_{,i}.$$

Show that the analogous expression can be written for an antisymmetric tensor of the second rank A^{ik}*:*

$$A^{ki}_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{ki})_{,i}.$$

[9]

Solution B1b. *[Seen similar]*

$$A^i_{;i} = A^i_{,i} + \Gamma^i_{in}A^n = A^i_{,i} + (\ln\sqrt{-g})_{,n}A^n$$

(see the previous sub-question). Taking into account that

$$(\ln\sqrt{-g})_{,n} = \frac{(\sqrt{-g})_{,n}}{(\sqrt{-g})},$$

one obtains

$$A^i_{;i} = A^i_{,i} + \frac{(\sqrt{-g})_{,n}}{(\sqrt{-g})}A^n = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}A^i_{,i} + (\sqrt{-g})_{,i}A^i\right) = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^i)_{,i}.$$

[4]

$$A^{ki}_{;i} = A^{ki}_{,i} + \Gamma^k_{in}A^{ni} + \Gamma^i_{in}A^{kn}.$$

Since $A^{ni} = -A^{in}$

$$\Gamma^k_{in}A^{ni} = -\Gamma^k_{in}A^{in} = -\Gamma^k_{ni}A^{in} = -\Gamma^k_{in}A^{ni}, \text{ hence } \Gamma^k_{in}A^{ni} = 0.$$

Thus

$$A^{ki}_{;i} = A^{ki}_{,i} + (\ln\sqrt{-g})_{,i}A^{ki}$$

(see the previous sub-question) and finally

$$A^{ki}_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{ki})_{,i}.$$

[5]

Question 9 (a) *Give brief explanation of what is meant by the limit of stationarity and the event horizon of a black hole and how to determine their locations. What is meant by ergosphere and where it is located?* [11]

Solution B2a. *[seen similar]*

The surface $g_{00} = 0$ (this equation determines the location of this surface) is called the limit of stationarity. No particle can be in rest inside this surface [but it does not mean that such a particle should move inward.]. Let us consider ds for the test particle in rest, i.e. put $dr = d\theta = d\phi = 0$, in this case

$$ds^2 = g_{00} dx^{0^2}.$$

If $g_{00} = 0$ then $ds^2 = 0$, which means that the world line of the particle at rest is the world line of light, hence at the surface $g_{00} = 0$ no particle with finite rest mass can be at rest. [4]

The surface $g^{11} = 0$ (this equation determines the location of this surface)is called the event horizon. No particle can move outward from inside this surface. Let us consider a surface $F(r) = const$ and let $n_i = F_{,i}$ is its normal. If $g^{11} = 0$ then $g^{ik} n_i n_k = g^{11} n_1 n_1 = 0$, which means that n_i is the null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon [within the event horizon all particles should move inward.] [5]

The ergosphere is the region outside the event horizon, where rotational energy of the black hole is located, that is why it is possible to extract the rotational energy of the Kerr black hole.The ergosphere is located between the limit of stationarity and the event horizon. [2]

(b) *Consider a rotating black hole described by the Kerr metric given in the rubric. Find the mass (express your result in solar masses) and angular momentum parameter of the black hole, $\alpha = 2a/r_g$, if its ergosphere in the equatorial plane ($\theta = \pi/2$) lies between $r_{min} = 125km$ and $r_{max} = 150km$.* [11]

Solution B2b. *[seen similar]*

Location of the event horizon corresponds to

$$g^{11} = 0.$$

Taking into account that all out of diagonal components $g_{1i} = 0$ (if $i \neq i$), one can see that $g^{11} = 1/g_{11}$ and the location of event horizon can be determined from $g_{11} = \infty$ or, as follows from the expressions for Kerr metric given in the rubric, from

$$\Delta = r^2 - r_g r + a^2 = 0.$$

There are two solutions

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 + 4a^2}}{2}.$$

The outer event horizon,r_{hor} corresponds to the sign "+", hence

$$r_{hor} = \frac{r_g}{2}\left(1 + \sqrt{1-\alpha^2}\right).$$

[4]

The location of limit of stationarity corresponds to

$$g_{00} = 0.$$

In the case of Kerr metric this corresponds to

$$1 - \frac{r_g r}{\rho^2} = 0,$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta,$$

hence from

$$r^2 - r_g r + a^2 \cos^2\theta = 0.$$

There are two solutions

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 + 4a^2\cos^2\theta}}{2}.$$

The outer limit of stationarity, r_{st} corresponds to the sign "+", hence

$$r_{st} = \frac{r_g}{2}\left(1 + \sqrt{1-\alpha^2\cos^2\theta}\right) = r_g = \frac{2GM}{c^2}$$

(because for the equatorial plane $theta = \pi/2$).Thus we have

$$r_{min} = \frac{GM}{c^2}\left(1 + \sqrt{1-\alpha^2}\right) \text{ and } r_{max} = \frac{2GM}{c^2}.$$

[4]

Hence $M/M_\odot = r_{max}/3\text{km} = 50$ and

$$\alpha = \sqrt{1 - \left(\frac{2r_{min}}{r_{max}} - 1\right)^2} = \frac{2}{r_{max}}\sqrt{r_{min}\left(r_{max} - r_{min}\right)} = \frac{2}{150}\sqrt{125 \times 25} = \frac{\sqrt{5}}{3} \approx 0.75.$$

[3]

Question 10 (a) *Prove the Bianchi identity.* [8]

Solution B3a. *[seen similar]*

The Bianchi identity, $R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0$, in the local galilean frame of reference, where all Christoffel symbols are equal to zero, can be re-written as

$$R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k} = 0$$

[2]

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and the Riemann tensor in this frame can be written as

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} = \Gamma^i_{km,l} - \Gamma^i_{kl,m},$$

[2]

$$R^n_{ikl,m}+R^n_{imk,l}+R^n_{ilm,k} = \left(\Gamma^n_{il,k} - \Gamma^n_{ik,l}\right)_{,m} + \left(\Gamma^n_{im,l} - \Gamma^n_{il,m}\right)_{,k} + \left(\Gamma^n_{ik,m} - \Gamma^n_{im,k}\right)_{,l} =$$

$$= \Gamma^n_{il,k,m} - \Gamma^n_{ik,l,m} + \Gamma^n_{im,l,k} - \Gamma^n_{il,m,k} + \Gamma^n_{ik,m,l} - \Gamma^n_{im,k,l} =$$

$$= [\Gamma^n_{il,k,m} - \Gamma^n_{il,m,k}] + [\Gamma^n_{ik,m,l} - \Gamma^n_{ik,l,m}] + [\Gamma^n_{im,l,k} - \Gamma^n_{im,k,l}] = [0] + [0] + [0] = 0.$$

[4]

(b) *Prove that the covariant Riemann tensor $R_{iklm} = g_{in}R^n_{klm}$ is antisymmetric in each of the index pairs i,k and l,m ($R_{iklm} = -R_{kilm} = -R_{ikml}$) and is symmetric under the interchange of two pairs with one another ($R_{iklm} = R_{lmik}$). Using these properties, show that by contracting the Bianchi identity on the pairs of indices i,k and l,n, one obtains that the covariant divergence of the Einstein tensor G^i_k (see rubric) is equal to zero.* [14]

Solution B3b. *[seen similar]*

In the local galilean frame of reference

$$R_{iklm} = \eta_{in}R^n_{klm} = \eta_{in}\left(\Gamma^n_{km,l} - \Gamma^n_{kl,m}\right) =$$

$$= \frac{1}{2}\eta_{in}\left[g^{np}(g_{kp,m} + g_{mp,k} - g_{km,p}\right]_{,l} - \frac{1}{2}\eta_{in}\left[g^{np}(g_{kp,l} + g_{lp,k} - g_{kl,p}\right]_{,m} =$$

$$= \frac{1}{2}\eta_{in}\eta^{np}\left(g_{kp,m,l} + g_{mp,k,l} - g_{km,p,l} - g_{kp,l,m} - g_{lp,k,m} + g_{kl,p,m}\right) =$$

$$= \frac{1}{2}\delta^p_i\left(g_{mp,k,l} - g_{km,p,l} - g_{lp,k,m} + g_{kl,p,m}\right) = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right).$$

[4]

$$R_{kilm} = \frac{1}{2}\left(g_{km,i,l} + g_{il,k,m} - g_{kl,i,m} - g_{im,k,l}\right) = -\frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) =$$
$$= -R_{iklm}.$$

[2]

$$R_{ikml} = \frac{1}{2}\left(g_{il,k,m} + g_{km,i,l} - g_{im,k,l} - g_{kl,i,m}\right) = -\frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) =$$
$$= -R_{iklm}.$$

[2]

$$R_{lmik} = \frac{1}{2}\left(g_{lk,m,i} + g_{mi,l,k} - g_{li,m,k} - g_{mk,l,i}\right) = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) =$$
$$= R_{iklm}.$$

[2]

After contracting the Bianchi identity we obtain

$$g^{kl}R^i_{klm;i} + g^{kl}R^i_{kil;m} + g^{kl}R^i_{kmi;l} = g^{in}g^{kl}\left(R_{nklm;i} + R_{nkil;m} + R_{nkmi;l}\right) =$$
$$= g^{kl}g^{in}\left(-R_{knlm;i} + R_{nkil;m} - R_{nkim;l}\right) = -g^{in}R_{nm;i} + g^{kl}R_{kl;m} - g^{kl}R_{km;l} = -R^i_{m;i} + R_{,m} - R^l_{m;l} =$$
$$= -2R^i_{m;i} + R_{,m} = 0.$$

[3]

Hence

$$G^i_{k;i} = \left(R^i_k - \frac{1}{2}\delta^i_k R\right)_{;i} = -\frac{1}{2}\left(-2R^i_k + R_{,k}\right) = 0.$$

[1]

Question 11 (a) *A weak gravitational wave is a small perturbation of the Minkowski metric, $g_{ik} = \eta_{ik} + h_{ik}$. Show that, to terms of first order in h_{ik}, the contravariant metric tensor is $g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}$. Consider a linear transformation $x^i = x'^i + \xi^i$, where ξ^i are small functions of x^i. Show that $h_{ik} = h'_{ik} - \xi_{i,k} - \xi_{k,i}$. Prove that it is always possible to find such ξ^i that the Ricci tensor takes the following simple form:*

$$R_{ik} = -\frac{1}{2}\eta^{lm}h_{ik,l,m}.$$

[14]

Solution B4a. *[seen similar]*

If $g_{ik} = \eta_{ik} + h_{ik}$, where h_{ik} are small, contravariant metric tensor can be written as $g^{ik} = \eta^{ik} + a^{ik}$, where a^{ik} are also small. Taking into account that $g_{ik}g^{kn} = \delta_i^n$ we have

$$(\eta_{ik} + h_{ik})(\eta^{kn} + a^{kn}) = \delta_i^n, \quad \delta_i^n + \eta_{ik}a^{kn} + h_{ik}\eta^{kn} = \delta_i^n,$$

$$\eta_{ik}a^{kn} = -h_{ik}\eta^{kn}, \quad \eta^{im}\eta_{ik}a^{kn} = -\eta^{im}h_{ik}\eta^{kn}, \quad \delta_k^m a^{kn} = -\eta^{im}\eta^{kn}h_{ik},$$

$$a^{mn} = -\eta^{mi}\eta^{nk}h_{ik}, \quad \text{finally} \quad g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}.$$

[3]

$$g_{ik} = \tilde{S}_i^n \tilde{S}_k^m g'_{nm}, \quad \text{where} \quad \tilde{S}_k^i = \frac{\partial x'^i}{\partial x^k} = \delta_k^i - \xi^i_{,k},$$

hence

$$\eta_{ik} + h_{ik} = (\delta_i^n - \xi^n_{,i})(\delta_k^m - \xi^m_{,k})(\eta_{nm} + h'_{nm});$$

to terms of first order in the h'_{ik}

$$h_{ik} = -\eta_{ik} + \delta_i^n[\delta_k^m(\eta_{nm} + h'_{nm}) - \xi^m_{,k}\eta_{nm}] - \xi^n_{,i}\delta_k^m\eta_{nm} = -\eta_{ik} + \eta_{ik} + h'_{ik} - \xi^m_{,k}\eta_{im} - \xi^n_{,i}\eta_{nk} =$$

$$= h'_{ik} - \xi_{i,k} - \xi_{k,i}.$$

[4]

Writing the Ricci tensors to terms of first order (in linear approximation) we have

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} = \frac{1}{2}\eta^{lm}\left(h_{im,k,l} + h_{km,i,l} - h_{ik,m,l} - h_{im,l,k} - h_{lm,i,k} + h_{il,m,k}\right) =$$

$$= -\frac{1}{2}\eta^{lm}h_{ik,m,l} + \frac{1}{2}\eta^{lm}\left(h_{km,i,l} - h_{lm,i,k} + h_{il,m,k}\right) =$$

$$= -\frac{1}{2}\eta^{lm}h_{ik,m,l} + \frac{1}{2}\left(h^l_{k,i,l} - h_{,i,k} + h^m_{i,m,k}\right), \quad \text{where} \quad h = h^l_l.$$

[4]

We have four arbitrary functions ξ, thus we can impose on h_{ik} four supplementary conditions: $h^k_{i,k} - 1/2h_{,i} = 0$,

$$R_{ik} = -\frac{1}{2}\eta^{lm}h_{ik,l,m} + \frac{1}{2}\left(\frac{1}{2}h_{,k,i} - h_{,i,k} + \frac{1}{2}h_{,i,k}\right) = -\frac{1}{2}\eta^{lm}h_{ik,l,m}.$$

[3]

(b) *Two bodies of equal mass, $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves, show that in order of magnitude, $h \sim (r_g/R)(r_g/cP)^{2/3}$, where R is the distance to the system and $r_g = \frac{2Gm}{c^2}$ is the gravitational radius.* [8]

Solution B4b. *[seen similar]*

To an order of magnitude (and omitting indices) we have

$$h \sim \frac{G}{c^4R}\ddot{D} \sim \frac{G}{c^4R}mr^2P^{-2}.$$

[3]

Taking into account that according to Newton low

$$P^{-2} \sim Gmr^{-3}, \text{ we have } r \sim (GmP^2)^{1/3},$$

[2]

---

hence

$$h \sim \frac{Gm}{c^4RP^2}\left(GmP^2\right)^{2/3} \sim \frac{r_g}{c^2RP^2}\left(r_gc^2P^2\right)^{2/3} \sim \frac{r_g}{R}\left(\frac{r_g}{cP}\right)^{2/3}.$$

[3]

End of Paper