Queen Mary
University of London

M. Sci. Examination by course unit 2009

MTH720U/MTHM033 Relativity and gravitation. SOLUTIONS

Duration: 3 hours

Date and time: xx xxx 2009, xxxxh

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 *Give the definition of a tensor with N contravariant and M covariant indices. What is the rank of the tensor and the number of independent components if $N = 2$ and $M = 3$. State the covariance principle and explain why according to this principle all physical equations should contains only tensors.*

Solution 1 *[Book work]*

This is the mixed tensor of $N + M$ rank defined as the object containing 4^{N+M} components $A^{i_1\, i_2\, i_3 \ldots i_N}_{j_1\, j_2\, j_3 \ldots j_M}$ which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

$$A^{i_1\, i_2\, i_3 \ldots i_N}_{j_1\, j_2\, j_3 \ldots j_M} = \underbrace{S^{i_1}_{m_1} S^{i_2}_{m_2} S^{i_3}_{m_3} \ldots S^{i_N}_{m_N}}_{N \text{ times}} \underbrace{\tilde{S}^{n_1}_{j_1} \tilde{S}^{n_2}_{j_2} \tilde{S}^{n_3}_{j_3} \ldots \tilde{S}^{n_M}_{j_M}}_{M \text{ times}} A'^{m_1\, m_2\, m_3 \ldots m_N}_{n_1\, n_2\, n_3 \ldots n_M},$$

where

$$S^l_m = \frac{\partial x^l}{\partial x'^m} \text{ and } S^l_m = \frac{\partial x'^l}{\partial x^m}.$$

This tensor of the fifth rank and contains in the most general case $4^5 = 1024$ components. [5]

The Principle of Covariance says: The shape of all physical equations should be the same in an arbitrary frame of reference. Otherwise the physical equations [being different in gravitational field and in inertial frames of reference] would have different solutions. Laws of transformations for tensors and only for tensors keep the the shape of equations unchanged after transformations of coordinates. [3]

Question 2 *Transformation from a local inertial (or local Galilean) frame of reference $x^i_($G) to some non-inertial frame x^i is given by the following transformation matrix:*

$$S^i_{(G)k} \equiv \frac{\partial x^i}{\partial x^i_{(G)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + A(x^m) & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $A(x^m) \neq -1$ is a function of the coordinates x^m. Show that the metric in the non-inertial frame of reference x^i has the following form

$$ds^2 = (dx^0)^2 - \frac{dx^1)^2}{(1+A)^2} - (dx^2)^2 - (dx^3)^2.$$

.

[Hint: Express first g^{ik} in terms of the matrix $S^i_{(G)k}$ and then calculate g_{ik} taking into account that g^{ik} and g_{ik} are reciprocal with respect to each other.]

Solution 2 *[Unseen]*

Let us first calculate g^{ik}. One can rewrite $S^i_{(G)k}$ as

$$S^i_{(G)k} = \delta^i_k + A\delta^i_1\delta^1_k.$$

Taking into account that

$$g^{ik}_{(G)} = \eta^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

we have

$$g^{ik} = S^i_{(G)n} S^k_{(G)m} \eta^{nm} = (\delta^i_n + A\delta^i_1\delta^1_n)(\delta^k_m + A\delta^k_1\delta^1_m)\eta^{nm} =$$

$$= [\delta^i_n\delta^k_m + A(\delta^i_n\delta^k_1\delta^1_m + \delta^k_m\delta^i_1\delta^1_n) + A^2\delta^i_1\delta^1_n\delta^k_1\delta^1_m]\eta^{nm} =$$

$$= \eta^{in} + A(\eta^{i1}\delta^k_1 + \eta^{1k}\delta^i_1) + A^2\eta^{11}\delta^i_1\delta^k_1 = \eta^{in} - A(\delta^i_1\delta^k_1 + \delta^k_1\delta^i_1) - A^2\delta^i_1\delta^k_1 = \eta^{in} - 2A\delta^i_1\delta^k_1 - A^2\delta^i_1\delta^k_1 =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+A)^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[5]

Determinant

$$|g^{ik}| = -(1 + A^2) \neq 0,$$

hence g_{ik} which is reciprocal to g^{ik},is presented by inverse matrix:

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+A)^{-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally

$$ds^2 = (dx^0)^2 - \frac{dx^1)^2}{(1+A)^2} - (dx^2)^2 - (dx^3)^2.$$

[3]

Question 3 *Given that the interval*

$$ds^2 = g_{ik}dx^i dx^k,$$

is a scalar. Prove that g_{ik} is a covariant tensor of the second rank. Show that without loss of generality this tensor has 10 independent components.

Solution 3. *[Seen similar]*

The fact that ds^2 is a scalar means

$$ds^2 = ds'^2, \;\; g_{ik}dx^i dx^k = g'_m n dx'^m dx'^n,$$

hence

$$0 = g_{ik}dx^i dx^k - g'_{mn}dx'^m dx'^n = g_{ik}dx^i dx^k - g'_{mn}\tilde{S}^m_i dx^i \tilde{S}^n_k dx^k = (g_{ik} - g'_{mn}\tilde{S}^m_i \tilde{S}^n_k)dx^i dx^k.$$

taking into account that dx^i and dx^k are arbitrary we conclude that the expression in brackets is equal to zero, hence

$$g_{ik} = g'_{mn}\tilde{S}^m_i \tilde{S}^n_k = \tilde{S}^m_i \tilde{S}^n_k g'_{mn},$$

thus according to the definition of the covariant second rank tensor g_{ik} indeed is the tensor of the second rank. [4]

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ik}dx^i dx^k}{2} = \frac{g_{ik}dx^i dx^k + g_{ik}dx^k dx^i}{2}.$$

The following substitution in the second term

$$i \to k \;\; k \to i$$

gives

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ki}dx^i dx^k}{2} = \frac{g_{ik} + g_{ki}}{2} dx^i dx^k = \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{g_{ik} + g_{ki}}{2}.$$

Obviously that

$$\tilde{g}_{ik} = \tilde{g}_{ki}.$$

We can use $\tilde{g}_{ik}$ instead g_{ik} and then changing notations just drop $\tilde{}$. [4]

Question 4 *Using the formulae for the Cristoffel symbol and covariant derivatives given in the rubric or otherwise, show that covariant derivatives of contrvariant metric tensor are equal to zero,* $g^{ik}_{;n} = 0$.

Solution 4. *[Book work]*

The shortest proof (otherwise) looks like this. Let us first prove that $Dg^{ik} = 0$. Let A_i is an arbitrary covariant vector. By the definition of D one can say that DA_i is also vector and its contravariant representation is

$$DA^i = g^{ik} DA_k.$$

On other hand

$$DA^i = D(g^{ik}A_k) = Dg^{ik}A_k + g^{ik}DA_k,$$

hence

$$g^{ik}DA_k = Dg^{ik}A_k + g^{ik}DA_k$$

which means that

$$Dg^{ik}A_k = 0$$

for arbitrary vector A_k, hence

$$Dg^{ik} = 0.$$

[5]

By definition of covariant derivatives

$$Dg^{ik} = g^{ik}_{;m} dx^m$$

for arbitrary infinitesimally small displacement dx^m which means that

$$g^{ik}_{;m} = 0.$$

[3]

[The proof with the help of expressions for Γ^i_{kn} *is approximately twice longer.]*

Question 5 *Using the Kerr metric given in the rubric, find the location of the outer event horizon, r_{hor}, and the outer limit of stationarity, r_{st}. Give a brief qualitative explanation what is the main difference between the limit of stationarity and the event horizon of a black hole.*

Solution 5. *[Book work]*

Location of the event horizon corresponds to

$$g^{11} = 0.$$

Taking into account that all out of diagonal components $g_{1i} = 0$ (if $i \neq i$), one can see that

$$g^{11} = \frac{1}{g_{11}}$$

and the location of event horizon can be determined from

$$g_{11} = \infty$$

or, as follows from the expressions for Kerr metric given in the rubric, from

$$\Delta = r^2 - r_g r + a^2 = 0.$$

There are two solutions

$$r_{\pm} = \frac{r_g \pm \sqrt{r_g^2 + 4a^2}}{2}.$$

The outer event horizon,r_{hor} corresponds to the sign "+", hence

$$r_{hor} = \frac{r_g}{2}\left(1 + \sqrt{1 - \alpha^2}\right),$$

where

$$\alpha = \frac{2a}{r_g} = \frac{ac^2}{GM}.$$

[4]

The location of limit of stationarity corresponds to

$$g_{00} = 0.$$

In the case of Kerr metric this corresponds to

$$1 - \frac{r_g r}{\rho^2} = 0,$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta,$$

hence from

$$r^2 - r_g r + a^2 \cos^2\theta = 0.$$

There are two solutions

$$r_{\pm} = \frac{r_g \pm \sqrt{r_g^2 + 4a^2 \cos^2\theta}}{2}.$$

The outer limit of stationarity, r_{st} corresponds to the sign "+", hence

$$r_{st} = \frac{r_g}{2}\left(1 + \sqrt{1 - \alpha^2 \cos^2\theta}\right).$$

[3]

Within the limit of stationarity no test particle can be in rest, but it does not mean that such a particle should move inward. Within the event horizon all particles should move inward. [1]

Question 6 *Show that the circle defined by $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity*

$$\frac{d\phi}{dt} = \Omega_{hor} = \frac{ac}{r_g r_{hor}}.$$

Solution 6. *[Unseen]*

Putting $dr = d\theta = 0$, $r = r_{hor}$ and $\theta = \pi/2$ (i.e. $\sin\theta = 1$, $\cos\theta = 0$ and $\rho = r$) into the Kerr metric, [1]
one obtains

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g rac}{\rho^2}\sin^2\theta d\phi dt = (1 - \frac{r_g}{r_{hor}})c^2dt^2 - (r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}})d\phi^2 + \frac{2r_g ac}{r_{hor}}d\phi dt =$$

[3]

$$= [(1 - \frac{r_g}{r_{hor}})c^2 - \Omega_{hor}^2(r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}}) + \frac{2r_g ac}{r_{hor}}\Omega_{hor}]dt^2 =$$

$$= [1 - \frac{r_g}{r_{hor}} - (\frac{a}{r_g r_{hor}})^2(r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}}) + \frac{2r_g a}{r_{hor}}\frac{a}{r_g r_{hor}}]c^2dt^2 =$$

$$= [1 - \frac{r_g}{r_{hor}} - (\frac{a}{r_g r_{hor}})^2(r_{hor} r_g + \frac{r_g a^2}{r_{hor}}) + \frac{2r_g a}{r_{hor}}\frac{a}{r_g r_{hor}}]c^2dt^2 = [1 - \frac{r_g}{r_{hor}} - (\frac{a^2}{r_g r_{hor}})(1 + \frac{a^2}{r_{hor}^2}) + \frac{2a^2}{r_{hor}^2}]c^2dt^2 =$$

$$= (1 - \frac{r_g}{r_{hor}} - \frac{a^2}{r_{hor}^2} + \frac{2a^2}{r_{hor}^2})c^2dt^2 = (1 - \frac{r_g}{r_{hor}} + \frac{a^2}{r_{hor}^2})c^2dt^2 = (r_{hor}^2 - r_g r_{hor} + a^2)\frac{c^2dt^2}{r_{hor}^2} = 0.$$

[3]

The fact that $ds^2 = 0$ means that this is the world line of a photon. [1]

Question 7 *The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as*

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that

$$g^{ik}p_i p_k = m^2c^2.$$

Then derive the Hamilton-Jacobi equation and explain how using this equation one can describe the motion of a test particle in given gravitational field.

Solution 7. *[Book work]*

From

$$ds^2 = g_i k dx^i dx^k$$

we have

$$g_{ik} u^i u^k = g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = \frac{g_{ik} dx^i dx^k}{ds^2} = \frac{ds^2}{ds^2} = 1.$$

After this we have

$$g^{ik} p_i p_k = p^k p_k = g_{kj} p^k p^j = m^2 c^2 g_{kj} u^k u^j = m^2 c^2.$$

[3]

For each covariant vector p_i we can find such a scalar that

$$p_i = -S_{,i}.$$

Substituting this to the previous formula we obtain the following equation for S:

$$g^{ik} S_{,i} S_{,k} = m^2 c^2.$$

This equation is called the Hamilton-Jacobi equation. [2]

The Hamilton-Jacobi equations "works" in the following way:

(i) We solve this single equation for single scalar field $S(x^m)$;

(ii) Taking partial derivatives we calculate covariant components of the four-momenta vector

$$p_i = -S_{,i};$$

(iii) With the help of g^{ik} we obtain contravariant components of the four-momenta vector

$$p^k = g^{ki} p_i;$$

(v) Then we calculate components of the four-velocity

$$u^i = \frac{p^i}{mc};$$

(vi) Finally we calculate world lines of test particles

$$x^i(s) = \int u^i ds.$$

[3]

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8

(a) *Using Bianchi identity prove that covariant divergence of the Ricci tensor R_{ik} tensor is related to the the gradient of the scalar curvature R by the following relationship:*

$$R^i_{k\,;\,i} = \frac{1}{2} R_{,k}.$$

[14]

Solution 8.a *[book work]*

After contracting the Bianchi identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0.$$

[1]

According to the definition of Ricci tensor

$$R^i_{kil} = R_{kl},$$

the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}.$$

[1]

Taking into account that the Riemann tensor is antisymmetric with respect permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km},$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}.$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip} R_{pklm;i},$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip} R_{pklm;i} = -g^{ip} R_{kplm;i} = g^{ip} R_{kpml;i}.$$

After all these manipulations we have

$$g^{ip} R_{kpml;i} + R_{kl;m} - R_{km;l} = 0.$$

[2]

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km}g^{ip}R_{kpml;i}+g^{km}R_{kl;m}-g^{km}R_{km;l}=\left(g^{km}g^{ip}R_{kpml}\right)_{;i}+\left(g^{km}R_{kl}\right)_{;m}-\left(g^{km}R_{km}\right)_{;l}=0.$$

[3]

In the first term expression in brackets can be simplified as

$$g^{km}g^{ip}R_{kpml}=g^{ip}R_{pl}=R^i_l.$$

In the second term expression in brackets can be simplified as

$$g^{km}R_{kl}=R^m_l.$$

[2]

According to the definition of scalar curvature

$$R=g^{km}R_{km},$$

the third term can be simplified as

$$\left(g^{km}R_{km}\right)_{;l}=R_{;l}=R_{,l}.$$

[2]

Thus

$$R^i_{l;i}+R^m_{l;m}-R_{,l}=0,$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i}-R_{,l}=0,\ \text{ or }\ R^i_{l;i}-\frac{1}{2}R_{,l}=0.$$

[3]

(b) *Using the EFEs given in the rubric and the identity proofed in the previous sub-question show that the covariant divergence of the stress-energy tensor is equal to zero, $T^i_{k\,;\,k}=0$.* [4]

Solution 8.b*[seen similar]*

Multiplying the EFEs

$$R_{ik}-\frac{1}{2}g_{ik}R=\frac{8\pi G}{c^4}T_{ik}$$

by g^{mk} we obtain

$$R^m_i-\frac{1}{2}\delta^m_i R=\frac{8\pi G}{c^4}T^m_k.$$

[2]

Taking covariant divergence of LHS and RHS of this equation we obtain

$$R^m_{i;\,m}-\frac{1}{2}\delta^m_i R_{;m}=\frac{8\pi G}{c^4}T^m_{k;\,m},$$

hence

$$T^m_{k;\,m} = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m}\right) = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2}R_{,i}\right) = 0.$$

[2]

(c) *The stress-energy tensor has the following form*

$$T_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix},$$

where ε is energy density and P is pressure (if $P > 0$) or tension (if $P < 0$). Using the Einstein equations express the scalar curvature in terms of ε and P. [4]

Solution 8.c[*Unseen*]

Contracting the EFEs written in mixed form (see the previous sub-question)we have

$$R^m_m - \frac{1}{2}\delta^m_m R = \frac{8\pi G}{c^4}T^m_m,$$

[2]

hence

$$R - \frac{4}{2}R = \frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(\varepsilon - 3P),$$

hence

$$R - 2R = \frac{8\pi G}{c^4}(\varepsilon - 3P),$$

finally

$$R = -\frac{8\pi G}{c^4}(\varepsilon - 3P).$$

[2]

Question 9

(a) *Using the equation $ds = 0$ with $\theta,\ \phi = const$, consider the propagation of radial light signals in the Schwarzschild space-time. Consider a photon emitted outward from $r = r_0$ at time $t = 0$. Show that the world-line of the photon is given by*

$$ct = r - r_0 + r_g \ln\frac{r - r_g}{r_0 - r_g}.$$

[5]

Solution 9.a [*Unseen*]

From $ds = 0$ for $\theta,\ \phi = const$, we have

$$c^2(1 - \frac{r_g}{r})dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 = 0,$$

[2]

hence

$$cdt = (1 - \frac{r_g}{r})^{-1}dr = r(r - r_g)^{-1}dr = \int r(r - r_g)^{-1}dr =$$

$$= \int (r - r_g + r_g)(r - r_g)^{-1}dr = (r - r_g) + r_g \ln(r - r_g) + C.$$

If at $t = 0$ $r = r_0$, then

$$C = -[(r_0 - r_g) + r_g \ln(r_0 - r_g)],$$

and finally

$$ct = r - r_0 + r_g \ln \frac{r - r_g}{r_0 - r_g}.$$

[3]

(b) *A particle moves along a radial geodesic in the Schwarzschild metric. Using the expression for ds and an appropriate component of geodesic equation, show that if the particle starts to fall freely from infinity, then*

$$r(\tau) = \left[r^{3/2}(\tau_0) - \frac{3}{2} c r_g^{1/2} (\tau - \tau_0) \right]^{2/3},$$

where τ is the proper time ($ds = cd\tau$). [7]

Solution 9.b *[Seen similar]*

A particle moves along radial geodesic in the Schwarzschild metric, then

$$\frac{cd^2t}{ds^2} + \Gamma^0_{00} c^2 (\frac{dt}{ds})^2 + 2\Gamma^0_{01} c \frac{dt}{ds}\frac{dr}{ds} + \Gamma^0_{11} (\frac{dr}{ds})^2 = 0.$$

$$\Gamma^0_{00} = \frac{1}{2} g^{00} (g_{00,0} + g_{00,0} - g_{00,0}) = 0,$$

$$\Gamma^0_{01} = \frac{1}{2} g^{00} (g_{00,1} + g_{10,0} - g_{01,0}) = \frac{1}{2} g^{00} \frac{dg_{00}}{dr} = \frac{1}{2} (1 - \frac{r_g}{r})^{-1} \frac{d(1 - \frac{r_g}{r})}{dr} = \frac{r_g}{2r^2} (1 - \frac{r_g}{r})^{-1},$$

$$\Gamma^0_{11} = \frac{1}{2} g^{00} (g_{10,1} + g_{10,1} - g_{11,0}) = 0,$$

[2]

so we have

$$\frac{d^2t}{ds^2} + + \frac{r_g}{r^2} (1 - \frac{r_g}{r})^{-1} \frac{dt}{ds}\frac{dr}{ds} = 0,$$

or

$$\frac{dt}{ds}(\frac{dt}{ds}) + (1 - \frac{r_g}{r})^{-1} \frac{dt}{ds}\frac{d}{ds}(1 - \frac{r_g}{r}) = (1 - \frac{r_g}{r})^{-1} \frac{dt}{ds}[\frac{dt}{ds}(1 - \frac{r_g}{r})] = 0,$$

hence

$$\frac{dt}{ds}(1 - \frac{r_g}{r}) = C.$$

[2]

At infinity $\frac{dt}{ds} = c^{-1}$, hence $C = c^{-1}$. Substituting this into eq. for ds, we have

$$1 = (1 - \frac{r_g}{r})c^2(1 - \frac{r_g}{r})^{-2}c^{-2} - (1 - \frac{r_g}{r})^{-1}(\frac{dr}{ds})^2,$$

$$1 - \frac{r_g}{r} = 1 - (\frac{dr}{ds})^2 \Rightarrow (\frac{dr}{d\tau}) = -c\sqrt{\frac{r_g}{r}},$$

we take "$-$" for falling objects, then

$$\frac{2}{3}r^{3/2}(\tau) - r^{3/2}(\tau_0) = -cr_g^{1/2}(\tau - \tau_0),$$

and finally

$$r(\tau) = [r^{3/2}(\tau_0) - \frac{3}{2}cr_g^{1/2}(\tau - \tau_0)]^{2/3}.$$

[3]

(c) *Using the coordinate transformations*

$$c\tau = ct + \int \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}$$

show that the Schwarzschild metric takes the form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r}dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Expressing r in terms of $R - c\tau$, demonstrate that the latter metric is non-stationary. What can be said about the true character of the Schwarzschild space-time metric at $r = r_g$? [10]

Solution 9.c *[Book work]*

By differentiating

$$cd\tau = cdt + \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad dR = cdt + \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}.$$

Subtracting the first from the second we have

$$dR - cd\tau = \frac{dr}{r - r_g}\left(\frac{r^{3/2}}{r_g^{1/2}} - r_g^{1/2} r^{1/2}\right) =$$

$$= \frac{r^{1/2} dr}{(r - r_g) r_g^{1/2}}(r - r_g) = \left(\frac{r}{r_g}\right)^{1/2} dr,$$

hence

$$dr = \left(\frac{r_g}{r}\right)^{1/2}(dR - cd\tau).$$

Subtracting the first multiplied by r/r_g from the second we have

$$\frac{r}{r_g}cd\tau - DR = cdt(\frac{r}{r_g} - 1),$$

hence

$$cdt = \frac{crd\tau - r_g dR}{r - r_g}.$$

[3]

Then substituting the expressions for dr and cdt into ds^2 in the Schwarzschild form we obtain

$$ds^2 = \frac{r - r_g}{r}\left(\frac{rcd\tau - r_g dR}{r - r_g}\right)^2 - \frac{r_g}{r - r_g}(dR - cd\tau)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \frac{1}{r - r_g}\left[\frac{1}{r}(rcd\tau - r_g dR)^2 - r_g(dR - cd\tau)^2\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \left[c^2 d\tau^2(r - r_g) - 2cdRd\tau(\frac{r_g r}{r} - r_g) - dR^2(\frac{r_g^2}{r} - r_g)\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

From

$$r^{1/2} dr = r_g^{1/2} d(R - c\tau)$$

we have

$$\frac{2}{3} r^{3/2} = C + r_g^{1/2}(R - c\tau),$$

then choosing the constant of integration $C = 0$ so that $r = 0 \;\rightarrow\; R - c\tau = 0$, we have

$$r = \left[\frac{3}{2} r_g^{1/2}(R - c\tau)\right]^{2/3}.$$

[4]

Finally, putting this into the metric in new coordinates we have

$$ds^2 = c^2 d\tau^2 - \left[\frac{2r_g}{3(R - c\tau)}\right]^{2/3} - \left[\frac{3}{2} r_g^{1/2}(R - c\tau)\right]^{4/3}(d\theta^2 + \sin^2\theta d\phi^2),$$

we can see that the metric depends on τ, which means that the gravitational field is non- stationary. [2]

We can see that there is no physical singularity at $r = r_g$. [1]

Question 10 *Consider the propagation of a photon in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzschild gravitational field.*

(a) *Derive the Eikonal equation*

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k} = 0$$

from the Hamilton-Jacobi equation or otherwise. [4]

Solution 10.a*[Book work]*

The Eikonal equation can be obtained from Hamilton-Jacobi equation by setting $m = 0$ and replacing S by Ψ with $\frac{\partial \Psi}{\partial t} = -\omega$, where ω is frequency of the light, and replacing the constant angular momentum L by an impact parameter $\rho = cL/\omega$. [4]

[To derive this equation otherwise takes approximately the half of a page.]

(b) *Given that the solution of the Eikonal equation can be written in the following form*

$$\Psi = -\omega t + \frac{\omega \rho}{c}\phi + \Psi_r(r),$$

where ω is the frequency of the photon and ϱ is its impact parameter, find a differential equation for Ψ_r and show that

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{dr}{cdt} = \sqrt{1 - \frac{\varrho^2}{r^2} + \frac{\varrho^2 r_g}{r^3}}.$$

[12]

Solution 10.b *[Seen similar]*

Taking $\theta = \pi/2$ we can write down the Eikonal equation in the Schwarzschild metric as

$$\left(1 - \frac{r_g}{r}\right)^{-1} \left(\frac{\partial \Psi}{c\partial t}\right)^2 - \left(1 - \frac{r_g}{r}\right)\left(\frac{\partial \Psi}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial \Psi}{\partial \phi}\right)^2 = 0.$$

[3]

Then putting

$$\Psi = -\omega t + \frac{\omega \rho}{c}\phi + \Psi_r(r),$$

we have

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{\omega^2}{c^2} - \left(1 - \frac{r_g}{r}\right)\left(\frac{d\Psi_r}{dr}\right)^2 - \frac{\rho^2 \omega^2}{c^2 r^2} = 0,$$

which is the usual differential equation for $\Psi_r(r)$. [2]

The radial component of the four wave vector k^i can be found as (λ is a arbitrary scalar parameter along the world line of photon)

$$k^1 = \frac{dr}{d\lambda} = g^{11}k_1 = -g^{11}\frac{d\Psi}{dr} = (1 - \frac{r_g}{r})\sqrt{\frac{\omega^2}{c^2}(1 - \frac{r_g}{r}^{-2} - \frac{\rho^2\omega^2}{c^2 r^2}(1 - \frac{r_g}{r})^{-1}} =$$

$$= \frac{\omega}{c}\sqrt{1 - \frac{\rho^2}{r^2}(1 - \frac{r_g}{r})} = \frac{\omega}{c}\sqrt{1 - \frac{\rho^2}{r^2} + \frac{\rho^2 r_g}{r^3}}.$$

[4]

On other hand

$$k^0 = \frac{cdt}{d\lambda} = g^{00}k_0 = -g^{00}\left(\frac{\partial \Psi}{c\partial t}\right) = g^{00}\frac{\omega}{c} = (1 - \frac{r_g}{r})^{-1}\frac{\omega}{c}.$$

Thus

$$\frac{dr}{cdt} = \frac{\frac{dr}{d\lambda}}{\frac{cdt}{d\lambda}} = \left(1 - \frac{r_g}{r}\right)\sqrt{1 - \frac{\rho^2}{r^2} + \frac{r_g\rho^2}{r^3}}.$$

[3]

(c) *Find the regions of possible motions on the $(r-\rho)$ diagram and show that the radius of the unstable stable circular orbit for photons corresponds to $\varrho = \frac{3\sqrt{3}}{2}r_g$ and $r = \frac{3}{2}r_g$.* [6]

Solution 10.c *[Seen similar]*

The limits of the radial motion (the turning points) are determined by the roots of the expression under the square root:

$$1 - \frac{\rho^2}{r^2} - \frac{r_g\rho^2}{r^3} = 0, \text{ hence } r^3 - \rho^2 r + \rho^2 r_g = 0,$$

thus

$$\rho^2 = \frac{r^3}{r - r_g} \text{ and } \rho = \pm\frac{r^{3/2}}{(r-r_g)^{1/2}}.$$

[2]

When $r \to \infty$, $\rho \to r$. When $r \to r_g$, $\rho \to \infty$. [1]

The curve $\rho(r)$ has a minimum corresponding to unstable circular orbit:

$$\frac{d\rho}{dr} = 0$$

gives

$$\frac{3r^2}{r - r_g} - \frac{r^3}{(r-r_g)^2} = 0 \text{ or } 3(r - r_g) - r = 0,$$

finally

$$r_* = \frac{3}{2}r_g$$

and

$$\rho_* = \left(\frac{3}{2}r_g\right)^{3/2}\frac{1}{(\frac{3}{2}-1)^{1/2}r_g^{1/2}} = \frac{3\sqrt{3}}{2}r_g.$$

[3]

Question 11 *Consider a plane gravitational wave propagating along the x-axis. All components of $h_{ik} = g_{ik} - \eta_{ik}$ vanish except $h_{22} = -h_{33} \equiv h_+$ and $h_{23} = h_{32} = h_\times$. Let two test particles be located in the $(y-z)$ plane and separated by the 3-vector $l^\alpha = (0, l_0\cos\theta, l_0\sin\theta)$.*

(a) *Show that the perturbation of the distance δl between the two particles in the gravitational wave varies as*

$$\delta l = l - l_0 = \frac{l_0}{2}(h_+\cos 2\theta + h_\times \sin 2\theta).$$

[4]

Solution 11.a[Seen similar]

$$l = \sqrt{-g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta} =$$
$$= \sqrt{-(\eta_{\alpha\beta} + h_{\alpha\beta})\Delta x^\alpha \Delta x^\beta} =$$
$$\sqrt{-\eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta - h_+(\Delta y^2 - \Delta z^2) - 2h_\times \Delta y \Delta x} =$$
$$= \sqrt{l_0^2 - h_+ l_0^2(\cos^2\theta - \sin^2\theta) - 2h_x l_0^2 \cos\theta \sin\theta} =$$
$$= l_0[1 - \frac{h_+}{2}\cos 2\Theta - \frac{h_\times}{2}\sin 2\Theta]$$

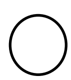
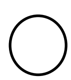
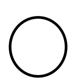
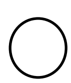
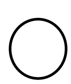
[3]

$$\delta l = l - l_0 = -\frac{1}{2} l_0 (h_+ \cos 2\theta + h_\times \sin 2\theta)$$

[1]

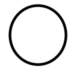
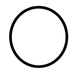
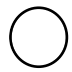
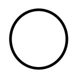
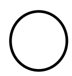
(b) *Consider a ring of test particles initially at rest in the $(y - z)$ plane and a plane monochromatic gravitational wave with frequency ω and polarization $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$. Sketch the shape of the ring perturbed by the gravitational wave at times $t = \frac{\pi}{2\omega}$, $\frac{\pi}{\omega}$, $\frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$. Repeat the analysis for a gravitational wave with another polarization: $h_+ = 0$, $h_\times \sin\omega(t - x/c)$. Finally consider the superposition of two polarized waves: $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = h_0 \cos\omega(t - x/c)$. What would you call this state of polarization?* [10]

Solution 11.b[Seen similar]

ωt	$\delta l(\theta)$	$l(\theta)$
0	0	
$\frac{\pi}{2}$	$-\frac{1}{2} l_0 h_0 \cos 2\theta$	
π	0	
$\frac{3\pi}{2}$	$\frac{1}{2} l_0 h_0 \sin\omega t \cos 2\theta$	
2π	0	

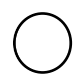
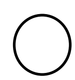
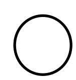
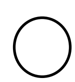
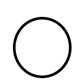
[3]

If $h_+ = 0$, $h_\times = h_0 \sin \omega t$ $\delta l(\theta) = -\frac{1}{2} l_0 h_0 \sin \omega t \sin 2\theta$

ωt	$\delta l(\theta)$	$l(\theta)$
0	0	
$\frac{\pi}{2}$	$-\frac{1}{2} l_0 h_0 \sin 2\theta$	
π	0	
$\frac{3\pi}{2}$	$\frac{1}{2} l_0 h_0 \sin 2\theta$	
2π	0	

[3]

If $h_+ = h_0 \sin \omega t$, $h_\times = h_0 \cos \omega t$
$\delta l(\theta) = -\frac{1}{2} l_0 h_0 (\sin \omega t \cos 2\theta + \cos \omega t \sin 2\theta) = -\frac{1}{2} l_0 h_0 (\sin \omega t + 2\theta) = -\frac{1}{2} l_0 h_0 \sin 2(\theta - \theta_0(t))$, where $\theta_0(t) = -\frac{1}{2}\omega t$

ωt	$\theta_0(t)$	$\delta l(\theta)$	$l(\theta)$
0	0	$-\frac{1}{2} l_0 h_0 \sin 2\theta$	
$\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{1}{2} l_0 h_0 \sin 2(\theta + \frac{\pi}{4}) = -\frac{1}{2} l_0 h_0 \cos 2\theta$	
π	$-\frac{\pi}{2}$	$-\frac{1}{2} l_0 h_0 \sin 2(\theta + \frac{\pi}{2}) = \frac{1}{2} l_0 h_0 \sin 2\theta$	
$\frac{3\pi}{2}$	$-\frac{3\pi}{4}$	$\frac{1}{2} l_0 h_0 \cos 2\theta$	
2π	$-\pi$	$-\frac{1}{2} l_0 h_0 \sin(2\theta + 2\pi) = -\frac{1}{2} l_0 h_0 \sin 2\theta$	

[3]

This polarization can be called circular polarization. [1]

(c) *Consider a binary system located in the center of our Galaxy ($R \approx 10kpc$), and consisting of two components of the same mass m. Show that to an order of* *magnitude the amplitude of the gravitational radiation generated by the binary and its frequency are $h_0 \sim r_g^2/(rR)$ and $\omega \sim (c r_g^{1/2} r^{-3/2})$ respectively, where r_g is the gravitational radius of each component and r is the separation between the two components. A future gravitational wave antenna detects gravitational radiation with frequency $10^{-3} Hz$ and amplitude 10^{-23}. Estimate the mass m and r.* [8]

Solution 11.c *[Unseen]*

Using quadruple formula and taking into account that in the binary

$$x = r\cos\omega t, \quad y = r\sin\omega t,$$

where

$$\omega = (\frac{GM}{r^3})^{\frac{1}{2}},$$

we have

$$h \sim \frac{2G}{3c^4 R}\omega^2 m r^2 \cos 2\omega t \sim \frac{2G}{3c^4 R}\frac{Gm}{r^3} m r^2 \cos 2\omega t \sim \frac{r_g^2}{rR}\cos 2\omega t.$$

Hence

$$h \sim \frac{r_g^2}{rR}, \quad \omega \sim (\frac{GM}{r^3})^{\frac{1}{2}} \sim c\frac{r_g^{\frac{1}{2}}}{r^{\frac{3}{2}}}, \quad \frac{r_g^2}{r} \sim hR.$$

[3]

$$\frac{r_g^{\frac{1}{2}}}{r^{\frac{3}{2}}} \sim \frac{\omega}{c}, \quad r \sim \frac{r_g^2}{hR}, \quad \frac{r_g^{\frac{1}{2}}}{r^3}(hR)^{\frac{3}{2}} \sim \frac{\omega}{c}, \quad r_g^{-\frac{5}{2}} \sim \frac{\omega}{c}(hR)^{-\frac{3}{2}}.$$

$$r_g = (\frac{\omega}{c})^{-\frac{2}{5}}(hR)^{\frac{3}{5}} = (\frac{\omega R}{c})^{-\frac{2}{5}} R h^{\frac{3}{5}} \sim (\frac{10^{-3}\cdot 10^4 \cdot 3 \cdot 10^18}{5 \cdot 3 \cdot 10^10})^{-\frac{2}{5}} \cdot 3 \cdot 10^4 \cdot 10^18 \cdot 10^{-\frac{22\cdot 3}{5}} =$$

$$= 10^{-\frac{18}{5}} \cdot 3 \cdot 10^22 \cdot 10^{-\frac{66}{5}} \approx 3 \cdot 10^{17-\frac{84}{5}} \approx 3 \cdot 10^{\frac{1}{5}} \approx 3 \cdot 1.4.$$

[3]

Hence $M = M_\odot(\frac{r_g}{3km}) \simeq 1.4 M_\odot$ and

$$\frac{r}{r_g} \approx (\frac{3km \cdot 10^{\frac{1}{5}}}{10^{-22}\cdot 3 \cdot 10^4 \cdot 10^{13} km}) \approx 10^5 \cdot 10^{\frac{1}{5}}.$$

$$r \simeq 5 \cdot 10^5 km$$

[2]

End of Paper