Queen Mary
University of London

M.Sc. EXAMINATION

MAS 412 (MTHM 033) Relativity and Gravitation

xxx, xx May 2008 xx:xx-xx:xx
Duration: 3 hours

SOLUTIONS

SECTION A

1. Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity. Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

SOLUTION A1 [book work]

This principle states that an uniform gravitational field is equivalent to a uniform acceleration of reference frame.

[1/8 Mark]

In Newton theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi,$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass. The fact that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1,$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

[2/8 Marks]

The General Relativity gives very simple and natural explanation of the Principle of Equivalence: In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in flat space-time when free particles move along straight lines which are geodesics in flat space-time.

[2/8 Marks]

The covariance principle says: The shape of all physical equations should be the same in an arbitrary frame of reference, including the most general case of non-inertial frames. If in contrast to the covariance principle the shape of physical equations were different in local inertial frames in presence of gravitational field and in non-inertial frames in absence of gravitational field then these equations would give different solutions, i.e. different predictions for (a) standing on the Earth, feeling the effects of gravity as a downward pull and (b) standing in a very smooth elevator that is accelerating upwards with the acceleration g , hence these equations would contradict to the basic postulate of the General Relativity, the principle of equivalence, which states that a uniform gravitational field (like that near the Earth) is equivalent to a uniform acceleration. Hence, the covariance principle is the mathematical formulation of the principle of equivalence.

[3/8 Marks]

2. Explain what is the reciprocal tensor. Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa. Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

SOLUTION A2 [book work]

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta^l_i.$$

[2/8 Marks]

We can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta^l_i.$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k.$$

[3/8 Marks]

We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1),$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

[3/8 Marks]

3. Give a rigorous proof that the interval squared,

$$ds^2 = g_{ik}dx^idx^k,$$

is a scalar if given that g_{ik}, the metric tensor, is a covariant tensor of the second rank. Prove that the metric tensor is symmetric.

 [This question continues overleaf . . .]

SOLUTION A3 [seen similar]

$$ds^2 = g_{ik} dx^i dx^k,$$

hence,

$$ds^2 = g_{ik} dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm} dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm} dx'^p dx'^w) = g'_{pw} dx'^p dx'^w = g'_{ik} dx'^i dx'^k = ds'^2,$$

thus

$$ds = ds'$$

which means that ds is a scalar.

[5/8 Marks]

$$ds^2 = g_{ik} dx^i dx^k = \frac{1}{2}(g_{ik} dx^i dx^k + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} dx^k dx^i + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik}) dx^i dx^k =$$

$$= \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}),$$

which is obviously symmetric one. Then we just drop "~".

[3/8 Marks]

4. A light signal emitted at the moment corresponding to time coordinate $x^0 + \Delta x^{0(1)}$ propagates from some point B with spatial coordinates $x^\alpha + \Delta x^\alpha$ to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + \Delta x^{0(2)}$. Given that $g_{0\alpha} = 0$, express the physical distance between A and B, l_{AB}, in terms of the metric tensor, g_{ik}, and Δx^α. You may assume that g_{ik} is the same in the points A and B.

SOLUTION A4 [seen similar]

For the proper time between any two events occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0.$$

[2/8 Marks]

Separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{00}(dx^0)^2.$$

The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero:

$$ds = 0.$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = -\frac{1}{g_{00}}\sqrt{-g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}$$

and

$$dx^{0(2)} = \frac{1}{g_{00}}\sqrt{-g_{\alpha\beta}g_{00})dx^\alpha dx^\beta},$$

hence

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{-g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}.$$

Then

$$dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)})$$

and finally

$$dl^2 = -g_{\alpha\beta}dx^\alpha dx^\beta,$$

and finally

$$l_{AB} = \int_B^A \sqrt{dl} = \sqrt{-g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta}.$$

[6/8 Marks]

5. Show that all covariant derivatives of metric tensor are equal to zero. Find the relationship between the Cristoffel symbols and first partial derivative of the metric tensor.

SOLUTION A5 [book work]

$$DA_i = g_{ik} DA^k$$
$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik},$$

hence

$$g_{ik}DA^k = g_{ik}DA^k + A^k Dg_{ik},$$

which obviously means that

$$A^k Dg_{ik} = 0.$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0.$$

Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

[3/8 Marks]

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km}\Gamma^m_{il},$$

we have

$$g_{ik;\, l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0.$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}.$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li},$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\,kl},$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

[5/8 Marks]

6. Explain what is the main difference between the limit of stationarity and the event horizon of a black hole?

SOLUTION A6 [book work]

The Limit of stationarity (Static Limit): the interval ds for test particle in rest

$$dr = d\theta = d\phi = 0.$$

In this case

$$ds^2 = g_{00} dx^{0^2},$$

We can see that if

$$g_{00} = 0,$$

then

$$ds^2 = 0,$$

which means that the world line of particle in rest is the world line of light. Hence, at the surface

$$g_{00} = 0$$

no particle with finite rest mass can be in rest. For this reason this surface is called the limit of stationarity.

[3/8 Marks]

Event Horizon is a spherically symmetric surface

$$F(r) = \text{const}.$$

Its normal vector is defined as usually as

$$n_i = F_{,i} = \delta^1_i \frac{dF}{dr}.$$

If at this surface

$$g^{11} = 0$$

then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = g^{11} \left(\frac{dF}{dr} \right)^2 = 0,$$

which means that n_i is a null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

[5/8 Marks]

7. Consider a rotating black hole described by the Kerr metric. Find the locations of event horizon, "limit of stationarity" and the "ergosphere"? (compare your results with the case of the Schwarzschild black hole).

SOLUTION A7 [**seen similar**]

Consider a rotating black hole described by the Kerr metric. Find the locations of event horizon, "limit of stationarity" and the "ergosphere"? (compare your results with the case of the Schwarzschild black hole). Describe briefly the Penrose process of extraction of energy from a rotating black hole and explain why this mechanism does not contradict to the statement, that nothing can escape from within black hole.

For the Kerr metric $g_{00} = 0$ gives

$$1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2 \cos^2\theta = 0,$$

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

[2/8 Marks]

The location of horizon in the Kerr metric: $g^{11} = 0$ $(g_{11} = \infty)$ corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

[2/8 Marks]

One can see easily that

$$r_{st} \geq r_{hor},$$

for example,

$$r_{st} = r_{hor}, \text{ if } \theta = 0, \text{ or } \theta = \pi \text{ (at the poles)},$$

and

$$r_{st} = 2r_g > r_{hor}, \text{ if } \theta = \frac{\pi}{2} \text{ (at the equator).}$$

The region between the limit of stationarity and the event horizon is called the "ergosphere".

[3/8 Marks]

In the Schwarzschild metric as one can see putting $a = 0$,

$$r_{hor} = r_{st},$$

which means that in this case the "ergosphere" does not exist.

[1/8 Marks]

SECTION B

Each question carries 22 marks. Only marks for the best TWO questions will be counted.

1. (a) **[10 Marks]** Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

SOLUTION B1(a) [Seen similar]

By definition the Ricci tensor is

$$R_{ik} = g^{lm}R_{limk} = R^l_{ilk},$$

where the curvature Riemann tensor is defined by

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}.$$

By straightforward calculations

$$A_{i;k;l} - A_{i;l;k} =$$

$$A_{i;k,l} - \Gamma^m_{li} A_{m;k} - \Gamma^m_{lk} A_{i;m} -$$

$$-A_{i;l,k} + \Gamma^m_{ki} A_{m;l} + \Gamma^m_{kl} A_{i;m} =$$

$$\left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \Gamma^m_{li}\left(A_{m,k} - \Gamma^n_{mk} A_n\right) -$$

$$-\left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} + \Gamma^m_{ki}\left(A_{m,l} - \Gamma^n_{ml} A_n\right) =$$

$$A_{i,k,l} - A_{i,l,k} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{il} A_{m,k} - \Gamma^m_{kl} A_{i,m} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{ik} A_{m,l} + \Gamma^m_{lk} A_{i,m} -$$

$$-\Gamma^m_{ik,l} A_m + \Gamma^m_{il}\Gamma^p_{mk} A_p + \Gamma^m_{kl}\Gamma^p_{im} A_p +$$

$$+\Gamma^m_{ik,l} A_m - \Gamma^m_{ik}\Gamma^p_{ml} A_p - \Gamma^m_{lk}\Gamma^p_{im} A_p =$$

$$= A_m\left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^p_{kl}\Gamma^m_{ip} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl} - \Gamma^p_{lk}\Gamma^m_{ip}\right) =$$

$$= A_m\left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl}\right),$$

hence

$$R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl},$$

and replacing k by l and l by k and then just putting $m = l$ we finally obtain

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

 [This question continues overleaf . . .]

(b) [**8 Marks**] Starting from the Einstein equations in the form

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik},$$

where G is the gravitational constant, prove that

$$T^i_k = \frac{c^4}{8\pi G} \left(R^i_k - \frac{1}{2} \delta^i_k R \right).$$

SOLUTION B1(b) [seen similar]

Contracting with g^{ik}, we have the Einstein equations in mixed form

$$R^i_k = \frac{8\pi G}{c^4} (T^i_k - \frac{1}{2} \delta^i_k T).$$

$$R = g^{ik} R_{ik} = \frac{8\pi G}{c^4} \left(g^{ik} T_{ik} - \frac{1}{2} g^{ik} g_{ik} T \right) = \frac{8\pi G}{c^4} \left(T^i_i - \frac{1}{2} \delta^i_i T \right) = \frac{8\pi G}{c^4} \left(T - \frac{1}{2} 4 \right) = -\frac{8\pi G}{c^4} T.$$

Thus

$$T = -\frac{c^4}{8\pi G} R.$$

Thus

$$T_{ik} = \frac{c^4}{8\pi G} \left(R_{ik} - \frac{1}{2} g_{ik} R \right),$$

then in mixed form we have

$$T^i_k = \frac{c^4}{8\pi G} \left(R^i_k - \frac{1}{2} \delta^i_k R \right).$$

(c) [**4 Marks**] c) What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

SOLUTION B1(c) [unseen]

This situation corresponds to gravitational fields (for example, gravitational waves), when the space-time is curved, but matter is absent (empty space-time).

2. The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2},$$

where L is the angular momentum and m is the mass of a particle, moving around Schwarzschild black hole.

(a) **[5 Marks]** What is the physical meaning of the "effective potential energy"? Explain how using U to find stable and unstable circular orbits.

SOLUTION B2(a)[book work]

The effective potential energy includes potential energy and that part of kinetic energy, which is related with non-radial, angular motion. Points at which $E = U$, (E is the conservative total energy) correspond to turning points, where $dr/dt = 0$.

$$U = E, \;\; U_r^{'} = 0,$$

corresponds to the circular orbit, stable, if $U_{rr}^{''} > 0$, and unstable, if $U_{rr}^{''} < 0$.

(b) **[10 Marks]** Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2r - 3r_g)^{1/2}\, r^{1/2}}.$$

SOLUTION B2(b)[seen similar]

Introducing $x = r_g/r$, we have $U_r^{'} = 0$ corresponds $U_x^{'} = 0$, so

$$[(1 - x)(1 + \alpha x^2]_x^{'} = 0,$$

where

$$\alpha = \frac{L^2}{m^2c^2r_g^2},$$

$$-1 - 3\alpha x^2 + 2\alpha x = 0,$$

and

$$\alpha = \frac{1}{x(2 - 3x)}.$$

Then

$$\frac{E^2}{m^2c^4} = (1 - x)(1 + \frac{x}{2 - 3x}) = \frac{2(1 - x)^2}{3 - 3x},$$

and finally

$$E = \frac{\sqrt{2}mc^2(1 - r_g/r)}{(2 - 3r_g/r)^{1/2}} = \frac{\sqrt{2}mc^2(r - r_g)}{(2r - 3r_g)^{1/2}r^{1/2}}.$$

 [This question continues overleaf . . .]

(c) **[7 Marks]** Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

SOLUTION B2(c)[unseen]

The last circular orbit corresponds the following system of equations: $E = U$, $U' = 0$, $U'' = 0$.

$$0 = U'' \sim 2\alpha(1 - 3x),$$

so $x = 1/3$, which corresponds to $r = 3r_g$.

$$\frac{E^2}{m^2c^4} = (1 - 1/3)(1 + 3/3^2) = 8/9,$$

and

$$E_{lo} = mc^2\frac{2\sqrt{2}}{3}.$$

Fraction of energy:

$$f = \frac{E_\infty - E_{lo}}{E_\infty} = 1 - \frac{2\sqrt{2}}{3} = 0.057$$

3. Consider a compact object of mass m moving along circular orbit around the black hole of mass M, assuming that $m \ll M$ and using the quadrupole formula for the metric perturbation associated with gravitational waves

(a) [**7 Marks**] Show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period;

SOLUTION B3(a) [seen similar]

$$x_1 = r\cos\omega_0 t,$$
$$x_2 = r\sin\omega_0 t,$$
$$D_{11} = mr_c^2(3\cos^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 + 3\cos 2\omega_0 t),$$
$$D_{22} = mr_c^2(3\sin^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 - 3\cos 2\omega_0 t),$$
$$D_{12} = \frac{3}{2}mr_c^2\sin 2\omega_0 t,$$

then

$$h_{11} = -\frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\cos 2\omega_0,$$
$$h_{22} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = -\frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$
$$h_{12} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\sin 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$

it is clear, that

$$\omega = 2\omega_0.$$

(b) [**9 Marks**] Show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

SOLUTION B3(b) [unseen]

From

$$r\omega_0^2 = \frac{GM}{r^2},$$

we have

$$\frac{1}{r^3} = \frac{\omega_0^2}{GM},$$

and finally

$$r_c^{-1} = (4GM)^{-1/3}\omega^{2/3}.$$

Thus

$$h \approx \frac{4\omega_0^2 Gmr^2}{c^4 R} = \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3}.$$

(c) [**6 Marks**] The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4}Hz < \omega < 3 \cdot 10^{-3}Hz$. From what distance will it be possible to detect gravitational radiation from the binary system, containing the black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$ around the massive black hole of mass $M = 10^3 M_\odot$?

SOLUTION B3(c) [unseen]

$$\omega_0^2 = \frac{GM}{r^3} = \frac{c^2}{2}\frac{2GM}{c^2 r^3} = c^2\frac{R_g}{2r^3},$$

hence,

$$\omega_0 = c\sqrt{\frac{R_g}{2r^3}} = c\sqrt{\frac{R_g}{2 \cdot 10^{12} R_g^3}} = \frac{10^{-6}c}{\sqrt{2}R_g} = \frac{10^{-4}Hz}{\sqrt{2}},$$

thus

$$\omega = 2\omega_0 = \sqrt{2}10^{-4}Hz \geq 10^{-4}Hz,$$

which means that the radiation is within LISA frequency range.

$$h = \frac{3 \cdot 10^5}{3 \cdot 10^{18}}\left(\frac{3 \cdot 10^5 \cdot 10^{-4}}{3 \cdot 10^{10}}\right)^{2/3} (\frac{m}{M.})(\frac{R}{1pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}$$

$$\approx 10^{-19}(\frac{m}{M.})(\frac{R}{1\ pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}.$$

Then

$$h = \frac{3 \cdot 10^5 cm}{R}(\frac{3 \cdot 10^5 \cdot 10^3 \cdot 1.4 \cdot 10^{-4}s^{-1}cm}{3 \cdot 10^{10}})^{2/3} > 10^{-23},$$

if

$$R < 3 \cdot 10^{23} \cdot 10^5 cm \cdot 10^{-4} \approx 1Mpc.$$

4. (a) [**8 Marks**] Derive the geodesic deviation equation

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm}u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

SOLUTION B4(a) [book work]

$$\eta^i = \frac{\partial x^i}{\partial v} \quad \delta v \equiv v^i \delta v \quad \frac{\partial u^i}{\partial v} = \frac{\partial v^i}{\partial s} \quad u^i = \frac{\partial x^i}{\partial s}$$

$$u^i_{;\,k}v^k = v^i_{;\,k}u^k$$

$$\frac{D^2 v^i}{ds^2} = (v^i_{;\,k}u^k)_{;l}\ \ u^l = (u^i_{;\,k}v^k)_{;}u^l = u^i_{;\,k;\,l}v^k u^l + u^i_{;\,k}v^k_{;\,l}u^l$$

$$\frac{D^2 v}{ds^2} = (u^i_{;\,l}u^l)_{;k} + u^m R^i_{mkl}u^k v^l, \quad u^i_{;\,l}u^l = 0$$

(b) [**9 Marks**] Consider two neighboring particles freely falling from rest in the Schwarzschild gravitational field in the same radial direction. Using the geodesic deviation equation show that the component of the Riemann tensor which is responsible for the tidal force in the radial direction is

$$R^1_{001} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

SOLUTION B4(b)[unseen]

Since two particles move in the same radial directions their spatial coordinates are r, θ, ϕ and $r + \Delta r, \theta, \phi$ respectively. Hence the separation vector $\eta^i = \delta^i_1 \Delta r(t)$. The fact that these two particles are in rest means that four- velocity of each particle is $u^i = \delta^i_0$,

hence, as follows from geodesic deviation equation

$$\frac{D^2\eta^1}{ds^2} = R^i_{mkl}u^k u^m \eta^l = R^i_{mkl}\delta^k_0\delta^m_0\delta^l_1\Delta r = R^1_{001}\Delta r.$$

$$R^1_{001} = \Gamma^1_{01,0} - \Gamma^1_{00,1} + \Gamma^1_{n0}\Gamma^n_{01} - \Gamma^1_{n1}\Gamma^n_{00}.$$

$$\Gamma^1_{01,0} = 0,$$

$$\Gamma^n_{01} = \frac{1}{2}g^{nm}\left(g_{0m,1} + g_{1m,0} - g_{01,m}\right) = \frac{1}{2}\delta^n_0 g^{00} g_{00,1},$$

$$\Gamma^n_{00} = \frac{1}{2} g^{nm} \left(2 g_{0m,0} + g_{1m,0} - g_{00,m}\right) = -\frac{1}{2} \delta^n_1 g^{11} g_{00,1},$$

hence

$$R^1_{001} = \frac{1}{2}\left(g^{11} g_{00,1}\right)_{,1} - \frac{1}{4} g^{00} g^{11} \left(g_{00,1}\right)^2 + \frac{1}{2} g^{11} g_{00,1} \Gamma^1_{11}.$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} g_{11,1}.$$

Taking into account that

$$f = g_{00} = 1 - \frac{r_g}{r}, \quad g_{00,1} = \frac{r_g}{r^2}, \quad g_{00,1,1} = -\frac{2r_g}{r^3}$$

and

$$g^{00} = -g_{11} = \frac{1}{g_{00}}, \quad g^{11} = g_{00},$$

we have

$$R^1_{001} = \frac{1}{2}(-ff')' + \frac{1}{4f} ff'^2 + \frac{1}{4}(-f)^2 f' (-\frac{1}{f})' = -\frac{1}{2}(ff'' + ff'^2 - f'^2) =$$

$$= -\frac{1}{2}\left[-2(1 - \frac{r_g}{r})\frac{r_g}{r^3} + (1 - \frac{r_g}{r} - 1)\frac{r_g^2}{r^4}\right] =$$

$$= \frac{r_g}{r^3}\left(1 - \frac{r_g}{r}\right) + \frac{r_g^3}{2r^5} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

(c) **[5 Marks]** If the height of an observer is $l \approx 2$m, find the radial distance $r \gg r_g$ from a solar mass neutron star at which the radial tidal 3-acceleration experienced by the observer at rest ($a = c^2 \frac{D^2 \eta^1}{ds^2}$) is equal to $100\ g \approx 10^3 \mathrm{ms}^{-2}$. You may assume that the observer's body is aligned along the radial direction and you may take the gravitational radius of the Sun to be 3 km.

SOLUTION B4(c) [unseen]

If $r \gg r_g$ we have

$$a \approx \frac{c^2 r_g l}{r^3} \approx 10^3 \mathrm{m\ s}^{-2},$$

hence

$$r \approx 0.1 \left(c^2 r_g \mathrm{s}^2 \mathrm{m}^{-1}\right)^{1/3} \approx 10^{-1}\left[\left(3 \cdot 10^8 \mathrm{m\ s}^{-1}\right)^2 \cdot 3 \cdot 10^3\ \mathrm{m} \cdot 2\ \mathrm{m\ m}^{-1}; \mathrm{s}^2\right]^{1/3} \approx$$

$$\approx 3 \cdot 10^{-1} \left(2 \cdot 10^19\right)^{1/3}\ \mathrm{m} \approx$$

$$\approx 800\ \mathrm{km}.$$

[End of examination paper.]