Queen Mary
University of London

M. Sci. Examination by course unit 2010

MTHM033/MTH720U Relativity and gravitation

Duration: 3 hours

Date and time: xx xxx 2010, xxxxh

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Polnarev

You are reminded of the following:

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ $\boldsymbol{N\ m^2\ kg^{-2}}$
Speed of light	c	$= 3 \times 10^{8}$ $\boldsymbol{m\ s^{-1}}$
1 kpc		$= 3 \times 10^{19}$ $\boldsymbol{m}$

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ ***and take on the values 1, 2, 3.***
Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ ***and take on the values 0, 1, 2, 3.***
The metric signature $(+ - - -)$ ***is used.***
Partial derivatives are denoted by ",".
Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m,$ ***where*** Γ^i_{kn} ***are Christoffel symbols***

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$u^i = dx^i/ds$ ***is the 4-velocity along the geodesic.***

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \ \ \textbf{\textit{where}} \ \ R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein tensor:

$$G_{ik} = R_{ik} - 1/2 g_{ik} R.$$

Einstein equations:

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where T^i_k is the Stress-Energy tensor.

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \frac{r_g}{r})} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\ \textbf{\textit{km}},\ \ \textbf{\textit{where}}\ M_\odot\ \textbf{\textit{is the mass of Sun}}.$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2}) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+ \frac{2 r_g r a c}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to source of gravitational radiation and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta}) dM$$

is the quadrupole tensor.

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 Prove that the metric tensor is symmetric. Explain how this symmetry and dimensions of space-time pre-determine the total number of the Einstein Field Equations required for the description of space-time geometry.

Question 2 Give the definition of the contravariant metric tensor g^{ik}. What manipulations with indices can be produced with the help of g_{ik} and g^{ik}? Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from a locally inertial frame of reference (local Galilean frame) to this non-inertial frame.

Question 3 Transformation from a local inertial (or local Galilean) frame of reference $x^i_{(0)}$ to some non-inertial frame x^i is given by the following transformation matrix: $S^i_{(0)k} = \delta^i_k + f\delta^i_0\delta^0_k$, where $f = f(x^m_{(0)})$ is a scalar field. Using the result of Question 1, show that the metric in the non-inertial frame of reference x^i has the following form: $ds^2 = (1+f)^{-2}(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.

Question 4 Explain why in order to prove that some tensor is identically equal to zero it is enough to show that all components of this tensor are equal to zero in the local galilean frame of reference. Then, prove that the Christoffel symbols, Γ^i_{kl}, are symmetric with respect to their low indices.

Question 5 Prove that all covariant derivatives of the metric tensor are equal to zero, - i.e., $g_{ik;l} = 0$. Then, using the symmetry of the Christoffel symbols proofed in question 4, show that the Christoffel symbols in terms of the metric tensor are $\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)$.

Question 6 Prove that the determinant of the metric tensor, g, is negative in all frames of reference. Then, prove the following identity:

$$2d\ln\sqrt{-g} = g^{ik}dg_{ik} = -g_{ik}dg^{ik}.$$

Question 7 Consider a light ray (electromagnetic signal) propagating in a gravitational field. The four-dimensional wave vector for the electromagnetic signal is defined as $k^i = dx^i/d\lambda$, where λ is some parameter varying along the ray. The scalar function Ψ is called the eikonal and defined as $k_i = \Psi_{,i}$. Derive the Eikonal equation (i.e., the equation for Ψ) and explain how using this equation one can describe the propagation of electromagnetic signals in a given gravitational field.

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8 .

(a) Prove the following identities:

$$\Gamma^i_{ik} - (\ln\sqrt{-g})_{,k} = 0 \ \text{ and } \ [\sqrt{-g}g^{ik})]_{,k} + \sqrt{-g}g^{kl}\Gamma^i_{kl} = 0.$$

[13]

(b) Prove that the covariant divergence of an arbitrary contravariant vector can be written as

$$A^i_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^i)_{,i}.$$

Show that the analogous expression can be written for an antisymmetric tensor of the second rank A^{ik}:

$$A^{ki}_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{ki})_{,i}.$$

[9]

Question 9 .

(a) Give brief explanation of what is meant by the limit of stationarity and the event horizon of a black hole and how to determine their locations. What is meant by ergosphere and where it is located? [11]

(b) Consider a rotating black hole described by the Kerr metric given in the rubric. Find the mass (express your result in solar masses) and angular momentum parameter of the black hole, $\alpha = 2a/r_g$, if its ergosphere in the equatorial plane ($\theta = \pi/2$) lies between $r_{min} = 125$km and $r_{max} = 150$km. [11]

Question 10 .

(a) Prove the Bianchi identity. [8]

(b) Prove that the covariant Riemann tensor $R_{iklm} = g_{in}R^n_{klm}$ is antisymmetric in each of the index pairs i,k and l,m ($R_{iklm} = -R_{kilm} = -R_{ikml}$) and is symmetric under the interchange of two pairs with one another ($R_{iklm} = R_{lmik}$). Using these properties, show that by contracting the Bianchi identity on the pairs of indices i,k and l,n, one obtains that the covariant divergence of the Einstein tensor G^i_k (see rubric) is equal to zero. [14]

Question 11 .

(a) A weak gravitational wave is a small perturbation of the Minkowski metric, $g_{ik} = \eta_{ik} + h_{ik}$. Show that, to terms of first order in h_{ik}, the contravariant metric tensor is $g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}$. Consider a linear transformation $x^i = x'^i + \xi^i$, where ξ^i are small functions of x^i. Show that $h_{ik} = h'_{ik} - \xi_{i,k} - \xi_{k,i}$. Prove that it is always possible to find such ξ^i that the Ricci tensor takes the following simple form:

$$R_{ik} = -\frac{1}{2}\eta^{lm}h_{ik,l,m}.$$

[14]

(b) Two bodies of equal mass, $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves, show that in order of magnitude, $h \sim (r_g/R)(r_g/cP)^{2/3}$, where R is the distance to the system and $r_g = \frac{2Gm}{c^2}$ is the gravitational radius. [8]

End of Paper