QUEEN MARY,

UNIVERSITY OF LONDON

M. Sc. Examination 2009

MTHM033/MTH720U Relativity and gravitation

Duration: 3 hours

Date and time: xx xxx 2009, xxxxh

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Do not start reading the question paper until instructed to do so.

The question paper must not be removed from the examination room.

 TURN OVER

You are reminded of the following:

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ N m^2 kg^{-2}
Speed of light	c	$= 3 \times 10^8$ m s^{-1}
1 kpc		$= 3 \times 10^{19}$ m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.
Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.
The metric signature $(+ - --)$ is used.
Partial derivatives are denoted by ",".
Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$$u^i = dx^i/ds \text{ is the 4-velocity along the geodesic.}$$

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \text{ where } R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.$$

Symmetry properties of the Riemann tensor:

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein equations:

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where T^i_k is the Stress-Energy tensor.

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \frac{r_g}{r})} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot) \text{ km}, \text{ where } M_\odot \text{ is the mass of Sun.}$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2}) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta) \sin^2\theta d\phi^2$$

$$+ \frac{2 r_g r a c}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Eikonal equation for photons:

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0,$$

where four-wave vector of the photon $k_i = -\frac{\partial \Psi}{\partial x^i}$.

 TURN OVER

Geodesic deviation equation:

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm}u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to source of gravitational radiation and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor.

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 Give the definition of a tensor with N contravariant and M covariant indices. What is the rank of the tensor and the number of independent components if $N = 2$ and $M = 3$. State the covariance principle and explain why according to this principle all physical equations should contains only tensors.

Question 2 Transformation from a local inertial (or local galilean) frame of reference $x^i_{(G)}$ to some non-inertial frame x^i is given by the following transformation matrix:

$$S^i_{(G)k} \equiv \frac{\partial x^i}{\partial x^i_{(G)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(x^m) & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & B(x^m) \end{pmatrix},$$

where $A(x^m)$ and $B(x^m)$ are some functions of the coordinates x^m. Show that the metric in the non-inertial frame of reference x^i has the following form

$$ds^2 = ???.$$

[Hint: Express first g^{ik} in terms of the matrix $S^i_{(G)k}$ and the calculate g_{ik} taking into account that g^{ik} and g_{ik} are reciprocal with respect to each other.]

Question 3 Given that the interval

$$ds^2 = g_{ik} dx^i dx^k$$

is a scalar, prove that g_{ik} is a covariant tensor of the second rank. Show that without loss of generality this tensor has 10 independent components.

Question 4 Using the formulae for the Cristoffel symbol and covariant derivatives given in the rubric or otherwise, show that covariant derivatives of contrvariant metric tensor are equal to zero, ${g^{ik}}_{;n} = 0$.

Question 5 Using the Kerr metric given in the rubric, find the location of the event horizon, r_{hor}, and the limit of stationarity, r_{st}. Compare these results with the case of a non-rotating black hole. Give brief qualitative explain of what is the main difference between the limit of stationarity and the event horizon of a black hole.

Question 6 Show that the circle defined by $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity,

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

Question 7 The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that

$$g^{ik} p_i p_k = m^2 c^2.$$

Then, derive the Hamilton-Jacobi equation and explain how using this equation one can describe the motion of a test particle in a given gravitational field.

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8 (a) Using Biancci identity prove that covariant divergence of the Ricci tensor R_{ik} is related to the the gradient of the scalar curvature R by the following relationship:

$$R^i_{k\,;\,i} = \frac{1}{2} R_{,k}.$$

[12]

(b) Using the Einstein Field Equations (EFEs) given in the rubric and the identity proved in the previous sub-question show that the covariant divergence of the stress-energy tensor is equal to zero, $T^i_{k\,;\,k} = 0$. Explain briefly why this equation is considered as energy and momenta conservation law. [13]

(c) Take the stress-energy tensor in the form

$$T^i_k = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix},$$

where ε is energy density and p is pressure (if $p > 0$) or tension (if $p < 0$). Using the Einstein equations, evaluate the scalar curvature in terms of ε and p. [13]

Question 9 (a) Using the equation $ds = 0$ with $\theta,\ \phi = \text{const}$, consider the propagation of radial light signals in the Schwarzschild space-time. Consider a photon emitted outward from $r = r_0$ at time $t = 0$. Show that the world-line of the photon is given by

$$ct = r - r_0 + r_g \ln \frac{r - r_g}{r_0 - r_g}.$$

[5]

(b) A particle moves along a radial geodesic in the Schwarzschild metric. Using the expression for ds and an appropriate component of geodesic equation, show that if the particle starts to fall freely from infinity, then

$$r(\tau) = \left[r^{3/2}(\tau_0) - \frac{3}{2} c r_g^{1/2} (\tau - \tau_0) \right]^{2/3},$$

where τ is the proper time ($ds = c d\tau$).

[10]

(c) Using the coordinate transformations

$$c\tau = ct + \int \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}\,(r - r_g)}$$

show that the Schwarzschild metric takes the form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Expressing r in terms of $R - c\tau$, demonstrate that the latter metric is non-stationary. What can be said about the true character of the Schwarzschild space-time metric at $r = r_g$?

[10]

Question 10 Consider the propagation of a photon in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzshild gravitational field.

(a) Derive the Eikonal equation

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0$$

from the Hamiltom-Jacobi equation or otherwise. [10]

(b) Given that the solution of the Eikonal equation can be written in the following form

$$\Psi = -\omega t + \frac{\omega \rho}{c} \phi + \Psi_r(r),$$

where ω is the frequency of the photon and ϱ is its impact parameter, find a differential equation for Ψ_r and show that

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{dr}{cdt} = \sqrt{1 - \frac{\varrho^2}{r^2} + \frac{\varrho^2 r_g}{r^3}}.$$

[10]

(c) Sketch the regions of possible motions on the $(r - \rho)$ diagram and hence show that the radius of the unstable stable circular orbit for photons corresponds to $\varrho = \frac{3\sqrt{3}}{2} r_g$ and $r = \frac{3}{2} r_g$. [10]

Question 11 Consider a plane gravitational wave propagating along the x-axis. All components of $h_{ik} = g_{ik} - \eta_{ik}$ vanish except $h_{22} = -h_{33} \equiv h_+$ and $h_{23} = h_{32} = h_\times$.

Let two test particles be located in the $(y-z)$ plane and separated by the 3-vector $l^\alpha = (0, l_0 \cos\theta, l_0 \sin\theta)$.

(a) Show that the perturbation of the distance δl between the two particles in the gravitational wave varies as

$$\delta l = l - l_0 = \frac{l_0}{2}(h_+ \cos 2\theta + h_\times \sin 2\theta).$$

[10]

(b) Consider a ring of test particles initially at rest in the $(y - z)$ plane and a plane monochromatic gravitational wave with frequency ω and polarization $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$. Sketch the shape of the ring perturbed by the gravitational wave at times $t = \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$. Repeat the analysis for a gravitational wave with another polarization: $h_+ = 0$, $h_\times \sin\omega(t - x/c)$. Finally consider the superposition of two polarized waves: $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = h_0 \cos\omega(t - x/c)$. What would you call this state of polarization? [10]

(c) Consider a binary system located in the center of our Galaxy ($R \approx 10kpc$), and consisting of two components of the same mass m. Show that to an order of magnitude the amplitude of the gravitational radiation generated by the binary and its frequency are $h_0 \sim r_g^2/(rR)$ and $\omega \sim (cr_g^{1/2}r^{-3/2})$ respectively, where r_g is the gravitational radius of each component and r is the separation between the two components. A future gravitational wave antenna detects gravitational radiation with frequency $10^{-3}Hz$ and amplitude 10^{-23}. Estimate the mass m and r. [10]

 END OF EXAMINATION