Queen Mary
University of London

# M.Sc. EXAMINATION

## MAS 412 Relativity and Gravitation

29 April 2008 10:00-13:00
Duration: 3 hours

*This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.*

*Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.*

*Don't turn over the page until instructed to do so by an invigilator.*

 

**You are reminded of the following:**

## PHYSICAL CONSTANTS

| | | |
|---|---|---|
| Gravitational constant | $G$ | $= 6.7 \times 10^{-11}$ N m$^2$ kg$^{-2}$ |
| Speed of light | $c$ | $= 3 \times 10^8$ m s$^{-1}$ |
| 1 kpc | | $= 3 \times 10^{19}$ m |

## NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+---)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

## USEFUL FORMULAS, which you may use without proof.

**Minkowski metric:**

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

**Covariant derivatives:**

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

**Christoffel symbols:**

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left( g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

**Geodesic equation:**

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$$u^i = dx^i/ds \text{ is the 4-velocity along the geodesic.}$$

**Riemann tensor:**

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \quad \text{where } R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.$$

**Symmetry properties of the Riemann tensor:**

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

**Bianchi identity:**

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

**Ricci tensor:**

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

**Scalar curvature:**

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

**Einstein equations:**

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where $T^i_k$ is the Stress-Energy tensor.

**Schwarzschild metric:**

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

**Gravitational radius**:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\ \text{km}, \quad \text{where } M_\odot \text{ is the mass of Sun.}$$

**Kerr metric:**

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+ \frac{2 r_g r a c}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where $J$ is the specific angular momentum.

**Eikonal equation for photons:**

$$g^{ik} \frac{\partial\Psi}{\partial x^i} \frac{\partial\Psi}{\partial x^k} = 0,$$

where four-wave vector of the photon $k_i = -\frac{\partial\Psi}{\partial x^i}$.

**Geodesic deviation equation:**

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m,$$

where $\eta^i$ is the 4-vector joining points on two infinitesimally close geodesics, and $u^k$ is the 4-velocity along the geodesic.

**Quadrupole formula for gravitational waves:**

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where $R$ is the distance to source of gravitational radiation and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor.

## SECTION A

*Each question carries 8 marks. You should attempt all questions.*

**1.** State the equivalence principle and explain the difference between interpretation of this principle in Newtonian theory and in General relativity. State the covariance principle and explain the relationship between this principle and the principle of equivalence.

**2.** What is the reciprocal tensor? Demonstrate how, using the reciprocal contravariant metric tensor $g^{ik}$ and the covariant metric tensor $g_{ik}$, you can form a contravariant tensor from covariant tensors and vice versa. Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0} S^k_{(0)0} - S^i_{(0)1} S^k_{(0)1} - S^i_{(0)2} S^k_{(0)2} - S^i_{(0)3} S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

**3.** Give a rigorous proof that the interval squared,

$$ds^2 = g_{ik} dx^i dx^k,$$

is a scalar if it is given that $g_{ik}$, the metric tensor, is a covariant tensor of the second rank. Prove that the metric tensor is symmetric.

**4.** A light signal emitted at the moment corresponding to time coordinate $x^0 + \Delta x^{0(1)}$ propagates from some point $B$ with spatial coordinates $x^\alpha + \Delta x^\alpha$ to a point $A$ with spatial coordinates $x^\alpha$ and then, after reflection at the moment corresponding to time coordinate $x^0$, the signal propagates back over the same path and is detected at point $B$ at the moment corresponding to time coordinate $x^0 + \Delta x^{0(2)}$. Given that $g_{0\alpha} = 0$, express the physical distance between $A$ and $B$, $l_{AB}$, in terms of the metric tensor, $g_{ik}$, and $\Delta x^\alpha$. You may assume that $g_{ik}$ is the same at points $A$ and $B$.

**5.** Show that all covariant derivatives of metric tensor are equal to zero. Find the relationship between the Christoffel symbols and first partial derivative of the metric tensor.

**6.** Explain the main difference between the limit of stationarity and the event horizon of a black hole?

**7.** Consider a rotating black hole described by the Kerr metric. Find the locations of event horizon, "limit of stationarity" and the "ergosphere"? Compare your results with the case of the Schwarzschild black hole.



*[Next section overleaf.]*

## SECTION B

*Each question carries 22 marks. Only marks for the best TWO questions will be counted.*

**1.** (a) [**10 Marks**] Give the definition of the Ricci tensor $R_{ik}$ and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

(b) [**8 Marks**] Starting from the Einstein equations in the form

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik},$$

where $G$ is the gravitational constant, prove that

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

(c) [**4 Marks**] What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while $R_{ikln}$ is not equal to zero?

**2.** The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2},$$

where $L$ is the angular momentum and $m$ is the mass of a particle, moving around a Schwarzschild black hole.

(a) [**5 Marks**] What is the physical meaning of the "effective potential energy"? Explain how $U$ can be used to find stable and unstable circular orbits.

(b) [**10 Marks**] Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows:

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2r - 3r_g)^{1/2}\, r^{1/2}}.$$

(c) [**7 Marks**] Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

**3.** Consider a compact object of mass $m$ moving along circular orbit around a black hole of mass $M$ assuming that $m \ll M$ and using the quadrupole formula for the metric perturbation associated with gravitational waves,

(a) [**7 Marks**] show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such a system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and $T$ is the orbital period;

(b) [**9 Marks**] show that, to an order of magnitude (omitting the indices $\alpha$ and $\beta$)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where $r_g$ is the gravitational radius of the mass $m$ and $R_g$ is the gravitational radius of the black hole.

(c) [**6 Marks**] The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4}Hz < \omega < 3 \cdot 10^{-3}Hz$. From what distance will it be possible to detect gravitational radiation from a binary system, containing a black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$, around a massive black hole of mass $M = 10^3 M_\odot$?



*[Next question overleaf.]*

**4.** (a) **[8 Marks]** Derive the geodesic deviation equation

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm}u^k u^l \eta^m,$$

where $\eta^i$ is the 4-vector joining points on two infinitesimally close geodesics and $u^k$ is the 4-velocity along the geodesic.

(b) **[9 Marks]** Consider two neighbouring particles freely falling from rest in the Schwarzschild gravitational field in the same radial direction. Using the geodesic deviation equation show that the component of the Riemann tensor which is responsible for the tidal force in the radial direction is

$$R^1_{001} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

(c) **[5 Marks]** If the height of an observer is $l \approx 2$m, find the radial distance $r \gg r_g$ from a solar mass neutron star at which the radial tidal 3-acceleration experienced by the observer at rest ( $a = c^2 \frac{D^2\eta^1}{ds^2}$) is equal to 100 $g \approx 10^3 \text{ms}^{-2}$. You may assume that the observer's body is aligned along the radial direction, and you may take the gravitational radius of the Sun to be 3 km.



*[End of examination paper.]*