Queen Mary
University of London

M.Sc. EXAMINATION

ASTMO41 Relativistic Astrophysics

21 May 2007 14:30-16:00
Duration: 1.5 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following.

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
Solar mass	$M_\odot$	2×10^{30} kg
Solar radius	$R_\odot$	7×10^5 km
1 kpc		3.09×10^{19} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, j, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulae

The following results may be quoted without proof

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}.$$

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Gravitational radius of body of mass M: $r_g = 2GM/c^2 = 3(M/M_\odot)$ km.

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta) \sin^2\theta d\phi^2 +$$

$$\frac{2 r_g r a c}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, $a = \frac{J}{Mc}$ and J is angular momentum.

For the Schwarzschild and Kerr metric: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

SECTION A

Each question carries 20 marks. You should attempt ALL questions.

1. A spacecraft exploring a planet of mass m and radius r moves around the planet along a circular orbit of radius $R = 2r$.

(a) Show that the redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft is $z \approx Gm/(2c^2 r)$.

[***10 Marks***]

(b) Given $r \approx 3 \times 10^4$ km and $z \approx 10^{-7}$, estimate the mean density of the planet.

[***10 Marks***]

2. A star forms a black hole of mass M.

(a) Show that to an order of magnitude its density at the moment immediately before the formation of the black hole is

$$2 \times 10^{19} \left(\frac{M}{M_\odot} \right)^{-2} \text{ kg m}^{-3}.$$

[***15 Marks***]

(b) For what mass is this equal to the density of water (10^3 kg/m^3)?

[***5 Marks***]

3. (a) In a Newtonian calculation, show that the escape velocity from the surface of a gravitating body is equal to the speed of light if the radius of the body is equal to its gravitational radius.

[***5 Marks***]

(b) In the same approximation calculate the maximum distance reached by a body thrown radially outward from the surface $r = r_g$ with a velocity $v < c$. Express the answer in terms of r_g and v. Discuss briefly how the answer would differ in a general relativistic calculation.

[***15 Marks***]

SECTION B

Each question carries 40 marks. Only marks for the best ONE question will be counted.

1. A supermassive black hole of mass M is surrounded by a stellar cluster, which consists of stars with mass m and radius r.

(a) Using simple Newtonian estimates, find the radius of tidal disruption, R_{TD}, in the gravitational field of the black hole.

[***5 Marks***]

(b) Find the critical black hole mass, M_{crit}, such that for $M < M_{\text{crit}}$ the tidal disruption takes place outside the black hole horizon. Express the answer in terms of m and r. Estimate M_{crit} if the cluster consists of giant stars with $m = 30M_{\odot}$ and $r = 10R_{\odot}$.

[***15 Marks***]

(c) Assume that the luminosity of AGNs and QSOs is generated by the disk accretion of gas onto a supermassive black hole and that the gas comes from the tidal disruption of stars. Assume for simplicity that the outer radius of this disk is R_{TD} and that its inner radius is $3r_g$. If the luminosity of the disk is proportional to its surface area and the geometry is Euclidean, show that the maximum of L is attained for $M = 3^{-9/4}M_{\text{crit}} \approx 0.08M_{\text{crit}}$.

[***20 Marks***]

2. (a) For a general black hole (not necessary Schwarzschild) explain why the surface where $g_{00} = 0$ is called the limit of stationarity. If g^{11} depends only on the radial coordinate, explain why the surface where $g^{11}(r) = 0$ is called the event horizon.

[***10 Marks***]

(b) Determine the position of the limit of stationarity, r_{ST}, and the position of the event horizon, r_H, for a Schwarzschild black hole and describe briefly the behaviour of the metric at $r = r_g$.

[***5 Marks***]

(c) Using the coordinate transformation

$$cd\tau = cdt + \frac{\sqrt{rr_g}}{r - r_g}dr, \quad dR = cdt + \frac{r\sqrt{r/r_g}}{r - r_g}dr,$$

show that the divergence of g_{11} at $r = r_g$ is related to the choice of the frame of reference rather than to a real physical singularity. Explain briefly why the previous frame of reference is inappropriate at $r = r_g$.

[***25 Marks***]

3. (a) Using the Kerr metric, find the surfaces corresponding to the limit of stationarity ($g_{00} = 0$) and the event horizon ($g^{11} = 0$). Compare this with the case of a Schwarzschild black hole.

[***20 Marks***]

(b) Show that the circle $r = r_g$ and $\theta = \pi/2$, is the worldline of a photon moving around the rotating black hole with angular velocity

$$\Omega = \frac{2ac}{r_g^2 + 2a^2}.$$

(Hint: put $dr = 0$, $d\theta = 0$ and $d\phi = \Omega dt$ into the Eq. for ds and show that $ds = 0$.)

[***15 Marks***]

(c) An observer moves with angular velocity Ω_{obs} along a circular orbit of radius r within the equatorial plane of a rotating black hole ($\theta = \pi/2$). Assuming that the metric tensor has diagonal form in the frame of reference comoving with the observer, show that

$$\Omega_{obs} = -\frac{a r_g r c}{r^2(r^2 + a^2) + r_g r a^2}.$$

[***10 Marks***]

[End of examination paper.]

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SECTION A.

A1.

a) The photon gravitational mass is $E/c^2 = h\nu/c^2$.
From energy conservation

$$h\nu - \frac{Gm}{R}\frac{h\nu}{c^2} = h\nu_s - \frac{Gm}{R}\frac{h\nu_s}{c^2},$$

where ν_s is the frequency of the photon at the surface of the planet.

[***5 Marks***] (*seen similar*)

Thus

$$\frac{\nu}{\nu_s} = \frac{1 - \frac{Gm}{rc^2}}{1 - \frac{Gm}{Rc^2}}.$$

Taking into account that in Newtonian limit $Gm/rc^2 \ll 1$,
for redshift z we have

$$1 + z = \frac{\lambda}{\lambda_s} = \frac{\nu_s}{\nu} \approx 1 - \frac{Gm}{Rc^2} + \frac{Gm}{rc^2},$$

hence

$$z = \frac{Gm}{rc^2}\left(1 - \frac{r}{R}\right) = \frac{Gm}{rc^2}(1 - \frac{1}{2}) = \frac{Gm}{2rc^2}.$$

[***5 Marks***] (*seen similar*)

b) From $m = (4\pi/3)\rho r^3$, where ρ is the mean density, we have

$$\rho = \frac{m}{\frac{4\pi}{3}r^3} = \frac{3m}{4\pi r^3}.$$

From a) we have

$$m = \frac{2rzc^2}{G},$$

hence

$$z = \frac{4\pi}{3}\frac{1}{2}\frac{G\rho r^2}{c^2} = \frac{2\pi G\rho r^2}{3c^2}.$$

[***5 Marks***] (*seen similar*)

Finally

$$\rho = \frac{3zc^2}{2\pi G r^2} = \frac{3 \times \left(3 \times 10^8 \text{ m s}^{-1}\right)^2 \times 10^{-7}}{2 \times 3.14 \times 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \times (3 \times 10^4 \text{ km})^2}$$

$$\approx \frac{3 \times 3 \times 3}{2 \times 3.14 \cdot 6.67} 10^{16-7+11-8-6} \frac{\text{m}^2 \text{ s}^{-2}}{\text{m s}^{-2} \text{ kg m}^2 \text{ kg}^{-2} \text{ m}^2} \approx 7 \times 10^3 \text{ kg} \cdot \text{m}^{-3}.$$

[***5 Marks***] (*seen similar*)

A2.

a) At the moment of BH formation the radius of the star is equal to its gravitational radius,

[*3* ***Marks***](*book work*)

hence to an order of magnitude

$$\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3} r_g^3} = \frac{3M}{4\pi \left(\frac{2GM}{c^2}\right)^3} = \frac{3M_\odot}{4\pi \left(\frac{2GM_\odot}{c^2}\right)^3} \left(\frac{M}{M_\odot}\right)^{-2} = \frac{3}{4\pi} \frac{M_\odot}{(3\ \mathrm{km})^3} \left(\frac{M}{M_\odot}\right)^{-2} =$$

[*7* ***Marks***](*book work*)

$$\frac{3 \cdot 2 \times 10^{30}\ \mathrm{kg}}{4 \cdot 3.14 \cdot 3^3 \times 10^9\ \mathbf{m}^3} \left(\frac{M}{M_\odot}\right)^{-2} = \frac{10^{21}}{10 \cdot 2 \cdot 3}\ \mathrm{kg}\cdot\ \mathrm{m}^{-3} \left(\frac{M}{M_\odot}\right)^{-2} \approx 2{\times}10^{19}\ \mathrm{kg}\cdot\ \mathrm{m}^{-3} \left(\frac{M}{M_\odot}\right)^{-2}.$$

[*5* ***Marks***] (*seen similar*)

b) If $\rho = 10^3\ \mathrm{kg} \cdot \mathrm{m}^{-3}$,

$$M = M_\odot \sqrt{\frac{2 \times 10^{19}}{10^3}} \approx 1.4 \times 10^8 M_\odot.$$

[*5* ***Marks***] (*seen similar*)

A3.

a) For escape velocity we have

$$\frac{v_{\text{esc}}^2}{2} - \frac{GM}{r} = 0,$$

hence

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}},$$

if

$$r = \frac{2GM}{c^2}, \quad v_{\text{esc}} = \sqrt{\frac{2GMc^2}{2GM}} = c.$$

[***5 Marks***] (*book work*)

b) If $v < v_{\text{esc}}$, the body being thrown up from the surface of gravitating object can reach the radius R, determined by

$$\frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{R},$$

hence

$$\frac{GM}{R} = \frac{GM}{r} - \frac{v^2}{2} = \frac{1}{2}(v_{\text{esc}}^2 - v^2),$$

[***5 Marks***] (*book work*) then

$$\frac{GM}{r}\frac{r}{R} = \frac{1}{2}(v_{\text{esc}}^2 - v^2),$$

or

$$\frac{r}{R}\frac{1}{2}v_{\text{esc}}^2 = \frac{1}{2}(v_{\text{esc}}^2 - v^2),$$

finally

$$R = r\frac{v_{\text{esc}}^2}{v_{\text{esc}}^2 - v^2}.$$

Let, for example, take $r = r_g$ (hence, $v_{\text{esc}} = c$) and $v = c/2$, in this case

$$R = r_g\frac{c^2}{c^2 - \frac{c^2}{4}} = \frac{4}{3}r_g > r_g.$$

[***5 Marks***] (*seen similar*) Thus in Laplacian version of BH the body can not reach infinity, but can reach some radius larger than r_g, while according to General Relativity any body initially at $r = r_g$ can move only inward with decreasing radius.

[***5 Marks***](*book work*)

SECTION B.

B1

a) To an order of magnitude gravitational force experienced by a particle of mass δm on the surface of the star from the star itself is $F_s \approx Gm\delta m/r^2$, while the tidal force producing a relative acceleration between the the same particle and the centre of the star to an order of magnitude is $F_{TD} \approx GM\delta mr/R^3$, hence defining the tidal radius as the radius at which $F_g \approx F_{TD}$, we have

$$\frac{Gm\delta m}{r^2} \approx \frac{GM\delta mr}{R_{TD}^3},$$

and finally, $R_{TD} \approx r(M/m)^{1/3}$.

[***5 Marks***] (*seen similar*)

b) For $M = M_{crit}$ from equality

$$R_{TD} = r_\odot \left(\frac{M}{M_\odot}\right)^{1/3} = r_g = \frac{2GM}{c^2} = 3\ \mathrm{km}\frac{M}{M_\odot},$$

we have

$$R_\odot \frac{r}{R_\odot}\left(\frac{M_{crit}}{M_\odot}\right)^{1/3}\left(\frac{m}{M_\odot}\right)^{-1/3} = \frac{2GM_\odot}{c^2}\frac{M_{crit}}{M_\odot},$$

hence

$$\left(\frac{M_{crit}}{M_\odot}\right)^{2/3} = \frac{R_\odot}{r_{g\odot}}\left(\frac{r}{R_\odot}\right)\left(\frac{m}{M_\odot}\right)^{-1/3},$$

[***5 Marks***] (*seen similar*) thus

$$\frac{M_{crit}}{M_\odot} = \left(\frac{R_\odot}{r_{g\odot}}\right)^{3/2}\left(\frac{r}{R_\odot}\right)^{3/2}\left(\frac{m}{M_\odot}\right)^{-1/2} \approx \left(\frac{7\times 10^5\ \mathrm{km}}{3\ \mathrm{km}}\right)^{3/2}\left(\frac{r}{R_\odot}\right)^{3/2}\left(\frac{m}{M_\odot}\right)^{-1/2} \approx$$

$$10^8\left(\frac{r}{R_\odot}\right)^{3/2}\left(\frac{m}{M_\odot}\right)^{-1/2}.$$

[***5 Marks***] (*seen similar*) For $m = 30M_\odot$ and $r = 10R_\odot$we have

$$M_{crit} \approx 10^9\frac{1}{3^{1/2}}M_\odot \approx 6.5\times 10^8 M_\odot.$$

[***5 Marks***] (*seen similar*)

c) Luminosity is

$$L = kS = k\pi[R_{TD}^2 - (3r_g)^2],$$

where k is the coefficient of proportionality between L and surface area S, hence

$$L = k\pi[R_{TD}^2(M_{crit})\left(\frac{M}{M_{crit}}\right)^{2/3} - 9r_g^2(M_{crit})\left(\frac{M}{M_{crit}}\right)^2].$$

[***5 Marks***] (*unseen*)

Taking into account that

$$R_{TD}(M_{crit}) = r_g(M_{crit}) = \frac{2GM_{crit}}{c^2},$$

we have

$$L \sim x^{2/3} - 9x^2,$$

where $x = M/M_{crit}$.

[***5 Marks***] (*unseen*)From

$$\frac{dL}{dx} \sim \frac{2}{3}x^{-1/3} - 18x = 0,$$

we have

$$\frac{2}{3}x^{-1/3} = 18x, \;\; x^{4/3} = \frac{1}{27}, \; x = 3^{-\frac{9}{4}} \approx 0.08.$$

Thus

$$M \approx 0.08 M_{crit}.$$

[***10 Marks***] (*unseen*)

B2

a) Let us consider a particle in rest,i.e. put $dr = d\theta = d\phi = 0$. In this case

$$ds^2 = c^2 g_{00} dt^2 = 0, \ if \ g_{00} = 0,$$

which corresponds to propagation of light.Hence in order to be at rest relative to the Schwarzschild frame of reference any particle with non-zero rest mass should move with the velocity of light relative to locally-inertial frame of reference, but this is impossible. In other words nothing can be at rest at the surface $g_{00} = 0$.

[***5 Marks***] (*book work*)

Let us consider a surface $F(r) =$ const and let

$$n_i = F_{,i} = \delta_i^1 \frac{dF}{dr}$$

is the outward normal to this surface. If $g^{11} = 0$, we have

$$g^{ik} n_i n_k = g^{ik} \delta_i^1 \delta_k^1 \left(\frac{dF}{dr}\right)^2 = g^{11} \left(\frac{dF}{dr}\right)^2 = 0,$$

which means that n_i directed outward is null vector. In other words, it can not be 4-velocity of any particle with non-zero rest mass. Thus no particle can move outward from such surface, that is why such a surface is called the event horizon.

[***5 Marks***] (*book work*)

b) Taking into account that the Schwarzschild metric is diagonal we have

$$g_{11} = \frac{1}{g^{11}} = -(1 - \frac{r_g}{r})^{-1} \to \infty, \text{ when } r \to r_g,$$

which means that the Schwarzschild metric is singular at $r = r_g$. On other hand

$$g_{00} = 1 - \frac{r_g}{r} \to 0, \text{ when } r \to r_g,$$

hence$r_{ST} = r_H = r_g$.

[***5 Marks***] (*book work*)

c) From the given transformation of coordinates we have

$$cd\tau - dR = \left(r^{1/2} r_g^{1/2} - r^{3/2} r_g^{-1/2}\right) \frac{dr}{r - r_g} = r^{1/2} r_g^{1/2} (1 - \frac{r}{r_g}) \frac{dr}{r - r_g} = -\frac{r^{1/2}}{r_g^{1/2}} \frac{(r - r_g)dr}{(r - r_g)} = -\frac{2}{3r_g^{1/2}} d(r^{3/2}),$$

hence

$$dr = \left(\frac{r_g}{r}\right)^{1/2} (dR - cd\tau)$$

and

$$c\tau - R = C - \frac{2}{3r_g^{1/2}} r^{3/2}.$$

Choosing the constant $C = 0$ so that $r = 0$ corresponds to $R = c\tau = 0$, we have

$$r = [\frac{3}{2} r_g^{1/2} (R - c\tau)]^{2/3},$$

[*10 Marks*] (*seen similar*)

then

$$cdt = cd\tau - \frac{r^{1/2} r_g^{1/2}}{r - r_g} \frac{r_g^{1/2}}{r^{1/2}} (dR - cd\tau) = cd\tau - \frac{r_g}{r - r_g} (dR - cd\tau) = \frac{(r/r_g) cd\tau - dR}{(r/r_g) - 1},$$

and

$$dR - cd\tau = \frac{\sqrt{r/r_g}}{r - r_g} (r - r_g) \, .$$

[*5 Marks*] (*seen similar*)

Substituting dr,cdt and r into the original metric we have

$$ds^2 = \left(1 - \frac{r_g}{r}\right) \frac{(cd\tau - \frac{r_g}{r})^2}{(1 - \frac{r_g}{r})^2} - \frac{r_g}{r} \frac{(dR - cd\tau)^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) =$$

$$\frac{1}{1 - \frac{r_g}{r}} \left[c^2 d\tau^2 \left(1 - \frac{r_g}{r}\right) + dR^2 \left(\frac{r_g^2}{r^2} - \frac{r_g}{r}\right) - 2cd\tau dR \left(\frac{r_g}{r} - \frac{r_g}{r}\right)\right] - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) =$$

$$c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

[*5 Marks*] (*seen similar*)

One can see that in this new frame of reference there is no singularity at $r = r_g$. This means that the divergence of g_{11} is related with bad choice of previous frame of reference rather than with any physical singularity. The previous frame of reference is bad at $r - r_g$, because it is formed by bodies in rest, which means that it is rigid, but gravitational forces are so strong when $r \to r_g$ that rigid frame of reference is impossible.

[*5 Marks*] (*book work*)

B3

a) For the Kerr metric the limit of stationarity is determined from $g_{00} = 0$.Thus we have $\rho^2 = r_g r_{ST}$ or

$$r_{ST}^2 - r_{ST} r_g + a^2 \cos^2\theta = 0.$$

So we have outer and inner limits of stationarity. The larger solution of this equation is

$$r_{ST} = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2 \cos^2\theta}.$$

[***5 Marks***] (*book work*)

For the Kerr metric the horizon is determined from $g^{11} = 0$.Thus we have $\Delta = 0$:

$$r_H^2 - r_g r_H = a^2 = 0.$$

So we have outer and inner horizons. The larger solution of this equation is

$$r_H = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

[***5 Marks***] (*book work*)

In Schwarzschild case, when $a = 0$ we have

$$r_{ST} = \frac{1}{2}(r_g + r_g) = r_g$$

and

$$r_H = \frac{1}{2}(r_g + r_g) = r_g,$$

hence $r_{ST} = r_H$.

[***5 Marks***] (*seen similar*)

b) For $dr = d\theta = 0$, $\theta = \pi/2$ and $d\phi = \Omega dt$

$$ds^2 = (1 - \frac{r_g}{r})c^2 dt^2 - (r^2 + a^2 + \frac{r_g a^2}{r})\Omega^2 dt^2 + \frac{2r_g ac}{r}\Omega dt^2 =$$

$$dt^2[c^2(1 - \frac{r_g}{r}) - (r^2 + a^2 + \frac{r_g a^2}{r})\Omega^2 + \frac{2r_g ac}{r}\Omega] = dt^2[-(r_g^2 + 2a^2)\Omega^2 + 2ac\Omega] =$$

$$-dt^2\Omega(r_g^2 + 2a^2)[\Omega - \frac{2ac}{r_g^2 + 2a^2}] = 0.$$

The fact that $ds = 0$ means that this is the world line of light.

[***10 Marks***] (*unseen*)

c) If the observer moves with angular velocity Ω_{obs} then in co-moving frame of reference

$$d\phi_{\text{obs}} = d\phi - \Omega_{obs} dt,$$

hence

$$ds^2 = g_{00}c^2 dt^2 + 2g_{03} c dt d\phi + g_{33} d\phi^2 =$$

$$g_{00}c^2 dt^2 + g_{33}\left(d\phi_{\text{obs}}^2 + 2\Omega_{obs} dt d\phi_{\text{obs}} + \Omega_{obs}^2 dt^2\right) + 2g_{03} c dt(d\phi_{\text{obs}} + \Omega_{obs} dt).$$

[***10 Marks***] (*seen similar*)

From the condition that the metric tensor is diagonal we have

$$2g_{03}c + 2g_{33}\Omega_{obs} = 0,$$

hence

$$\Omega_{obs} = -\frac{cg_{03}}{g_{33}} = -\frac{ar_g rc}{r^2(r^2 + a^2) + r_g ra^2}.$$

[***5 Marks***] (*unseen*)