Queen Mary
University of London

M.Sc. EXAMINATION

ASTMO41 Relativistic Astrophysics

15 May 2009 14:30-16:30
Duration: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following:

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^{8} m s^{-1}
Solar mass	$M_\odot$	2×10^{30} kg
Solar radius	$R_\odot$	7×10^{5} km
1 pc		3.1×10^{16} m
1 AU		1.5×10^{11} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, j, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulae

The following results may be quoted without proof

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Gravitational radius of body of mass M: $r_g = 2GM/c^2 = 3(M/M_\odot)$ km.

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 -$$

$$-(r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta) \sin^2\theta d\phi^2 + \frac{2 r_g r a c}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, $a = \frac{J}{Mc}$ and J is angular momentum.

For the Schwarzschild and Kerr metric: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

The eikonal equation in a gravitational field:

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0,$$

where $\Psi = -\int k_i dx^i$, $k_i = g_{ik} k^i$, $k^i = \frac{dx^i}{d\lambda}$ and λ is an arbitrary scalar parameter.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R} \frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to the source and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) dM \text{ is the quadrupole tensor.}$$

SECTION A

Each question carries 20 marks. You should attempt ALL questions.

1. A spacecraft observing a planet of average density ρ and radius r moves around the planet on a circular orbit of radius $R = 3r$.

(a) Using the energy conservation law, calculate the gravitational redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft. Give your answer in terms of ρ and m.

[***13 Marks***]

(b) Given that $\rho \approx 5$ g cm^{-3} and $z \approx 5 \times 10^{-10}$, estimate the mass of the planet.

[***7 Marks***]

2. As a result of gravitational collapse a cloud of mass M, initial density ρ_0 and initial radius R_0 forms a black hole. The density of the cloud at the moment the black hole forms, $\rho_{BH} = 10^4$ kg m^{-3}, is α times larger than its initial density, ρ_0.

(a) Assuming conservation of mass, show that the gravitational radius of the black hole is

$$r_g = c\sqrt{\frac{3}{8\pi G\rho_{BH}}}.$$

[***10 Marks***]

(b) Given that $\alpha = 10^9$, estimate in order of magnitude M, ρ_0 and R_0.

[***10 Marks***]

3. Consider a rotating black hole of mass M and angular momentum J.

(a) Explain briefly what is meant by the ergosphere.

[***5 Marks***]

(b) Calculate the ratio, f, of the outer radius of the ergosphere to the inner radius in the equatorial plane of the black hole. Express your result in terms of M and J.

[***15 Marks***]

SECTION B

Each question carries 40 marks. Only marks for the best ONE question will be counted.

1. A supermassive black hole of mass M is surrounded by a stellar cluster, which consists of stars of average density ρ_s.

(a) Using simple Newtonian estimates, find the radius of tidal disruption, R_{TD}, in the gravitational field of the black hole. Express your answer in terms of M and ρ_s.

[***6 Marks***]

(b) Find the critical mass, M_{crit}, as a function of ρ_s, such that black holes with $M > M_{crit}$ swallow stars without any tidal disruption. Estimate M_{crit} if the cluster consists of stars with $m = 1M_\odot$ and $r = 1R_\odot$.

[***14 Marks***]

(c) Assume that the luminosity of AGNs and QSOs is generated by the accretion of gas onto a supermassive black hole and that the gas comes from the tidal disruption of stars. Further assume for simplicity that: (i) the luminosity, L, is proportional to the total gas mass within the volume between the sphere of radius R_{TD} and the sphere of radius $3r_g$ (corresponding to the last stable circular orbit around a Schwarzschild black hole); (ii) the density of gas in this volume falls with distance from the centre of the cluster as $\rho_g(R) \propto R^{-2}$; (iii) the deviation of space from Euclidean geometry can be neglected. Show that L as a function of M has a maximum, and estimate the mass M_{max} for which this maximum is attained. Express your result in terms of M_{crit}.

[***20 Marks***]

2. Consider the motion of a particle in the gravitational field of a Schwarzschild black hole.

(a) A test particle moves along a circular orbit of radius r around the black hole. Using the Hamilton-Jacobi equation, show that the energy of the particle E is given by

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - U_{\rm eff}^2},$$

where $U_{\rm eff}$ is the "effective potential energy":

$$U_{\rm eff}(r) = mc^2\sqrt{\left(1-\frac{r_g}{r}\right)\left(1+\frac{L^2}{m^2c^2r^2}\right)},$$

 [This question continues overleaf ...]

L being the angular momentum and m the mass of the particle.

[***17 Marks***]

(b) Give a brief physical description of the "effective potential energy". Explain how U_{eff} can be used to find stable and unstable circular orbits. Eliminating the angular momentum L, express the energy E in terms of the circular orbit radius.

[***15 Marks***]

(c) Evaluate the radius of the innermost stable circular orbit and the energy of a particle moving along this orbit.

[***8 Marks***]

3. (a) Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

[***5 Marks***]

(b) Show that all covariant derivatives of the metric tensor are equal to zero.

[***9 Marks***]

(c) Consider a mass m moving on a circular orbit around a black hole of mass M, assuming that $m \ll M$. Using the quadrupole formula given in the rubric, show that all the amplitudes $h_{\alpha\beta}$ of the gravitational waves emitted by such a system are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$ and T is the orbital period. Show also that to an order of magnitude the amplitudes $h_{\alpha\beta}$ have the value

$$h \approx \frac{r_g}{R} \left(\frac{R_g \omega}{c} \right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

[***26 Marks***]

[End of examination paper.]