UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Physics 3C26: Quantum Mechanics

COURSE CODE	:	**PHYS3C26**
UNIT VALUE	:	**0.50**
DATE	:	**05-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

06-C1175-3-140

TURN OVER

Answer ALL SIX questions from Section A and TWO questions from Section B.

The numbers in square brackets in the right hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

The following results may be used:

In spherical polar co-ordinates, $d\mathbf{r} = r^2 \sin\theta \, dr d\theta d\phi$, and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

If n is a positive integer,

$$\int_0^\infty e^{-\alpha r} r^n dr = \frac{n!}{\alpha^{n+1}}.$$

The spin angular momentum operator $\hat{\mathbf{S}}$ is given by $\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\sigma}$ where $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ and, in the basis formed by $|\alpha>$ and $|\beta>$, the Pauli spin matrices are

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$|\alpha> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\beta> = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

SECTION A

1. Explain what is meant by an **observable** in quantum mechanics. [2]

   When does a measurement of the expectation value of an observable correspond to an eigenvalue of the observable? [1]

   An observable $\hat{Q}$ has just two orthonormal eigenstates ϕ_1 and ϕ_2 corresponding to eigenvalues -1 and $+1$ respectively. A system is described by a wave function ψ

   $$\psi = c_1\phi_1 + c_2\phi_2,$$

   which is normalised to unity.
   What is the probability of a measurement of $\hat{Q}$ giving the result $+1$? [1]

   What is the expectation value of $\hat{Q}$ before any measurement is made? [2]

2. The variance of an observable $\hat{Q}$ in a state with wave function Ψ is defined by

$$(\Delta\hat{Q})^2 = < (\hat{Q} - < \hat{Q} >)^2 > .$$

Explain the meaning of the terms in this expression. [3]

Show that the variance can also be expressed as

$$(\Delta\hat{Q})^2 = < \hat{Q}^2 > - < \hat{Q} >^2$$ [3]

3. Determine a real positive constant N such that the spherically symmetric three dimensional wave function

$$\psi(\mathbf{r}) = N\ \mathrm{e}^{-r}$$

is normalised to unity. [6]

4. The conceptual problems of quantum mechanics may be grouped under the three headings of **determinism** , **locality** and **reduction** of the wave function. Very briefly outline the problems in each of these three areas. [8]

5. What is meant by a **stationary state** in quantum mechanics? [2]

The time rate of change of the expectation value of an observable $\hat{A}$, for a system with wave function Ψ, is given by

$$\frac{\partial < \hat{A} >}{\partial t} = \frac{i}{\hbar}\int \Psi^*[\hat{H}, \hat{A}]\Psi d\tau + < \frac{\partial \hat{A}}{\partial t} > .$$

Explain the meaning of the terms on the right hand side of this equation and the significance of this result. [4]

6. Using Dirac notation, express

- The fact that a complete set of states $|u_n>$ are orthonormal [2]
- That any state $|x>$ may be expanded in terms of the $|u_n>$ [2]
- The coefficients of the expansion of $|x>$ in terms of the $|u_n>$ [2]
- That an operator $\hat{A}$ is Hermitian [2]

SECTION B

7. (a) A Hermitian operator $\hat{A}$ has a complete set of orthonormal eigenfunctions $|n>$. Show that in the basis $|n>$, the matrix representing $\hat{A}$ is diagonal with elements $A_{nn'} = a_n \delta_{nn'}$ where a_n is the eigenvalue of $\hat{A}$ corresponding to eigenvector $|n>$. [3]

(b) If $\hat{\mathbf{J}}$ is a quantum mechanical angular momentum operator, $\hat{J}^2$ and $\hat{J}_z$ satisfy the commutation relation $[\hat{J}^2, \hat{J}_z] = 0$. What does this tell us about these two observables? [2]

If $\hat{J}_+$ and $\hat{J}_-$ are defined by

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \quad ; \quad \hat{J}_- = \hat{J}_x - i\hat{J}_y,$$

show that

$$[\hat{J}_z, \hat{J}_+] = \hbar \hat{J}_+$$ [4]

(c) Given that, in addition,

$$[\hat{J}_z, \hat{J}_-] = -\hbar \hat{J}_-$$

and if, also

$$\hat{J}_z|j, m> = m\hbar|j, m>$$

show that $\hat{J}_+|j, m>$ is an eigenfunction of $\hat{J}_z$ corresponding to the eigenvalue $(m+1)\hbar$. [4]

(d) A particle has total spin quantum number $s = 1$. If the spin operator is $\hat{\mathbf{S}}$, what are the eigenvalues of $\hat{S}_z$ and $\hat{S}^2$? [2]

Write down the matrices of $\hat{S}_z$ and $\hat{S}^2$ in the basis formed from the normalised eigenstates $|s, m>$ of $\hat{S}^2$ and $\hat{S}_z$. [3]

(e) Verify that they commute. [2]

(f) Given that

$$\hat{S}_\pm|s, m> = \hbar\sqrt{s(s+1) - m(m \pm 1)}|s, m \pm 1>$$

find the matrix of $\hat{S}_x$ in the same basis. [6]

(g) Verify that the eigenvalues of $\hat{S}_x$ are the same as those of $\hat{S}_z$. [4]

8. (a) The Hamiltonian operator $\hat{H}$ for a one-dimensional harmonic oscillator of mass m and angular frequency ω is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

The operators $\hat{a}_+$ and $\hat{a}_-$ are defined by

$$\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(\hat{p} \pm im\omega\hat{x}),$$

where $\hat{p}$ and $\hat{x}$ are momentum and position operators satisfying $[\hat{x}, \hat{p}] = i\hbar$.
Show that

$$\hat{H} = (\hat{a}_+\hat{a}_- + \frac{1}{2})\hbar\omega. \qquad [5]$$

and

$$[\hat{a}_-, \hat{a}_+] = 1. \qquad [5]$$

(b) Assuming without proof that

$$[\hat{H}, \hat{a}_\pm] = \pm\hat{a}_\pm\hbar\omega$$

and using the notation $\hat{H}|n >= E_n|n >$, show that

$$\hat{H}\hat{a}_+|n >= (E_n + \hbar\omega)\hat{a}_+|n > \qquad [5]$$

(c) What is the interpretation of this result? [2]

(d) Given also that,

$$\hat{H}\hat{a}_-|n >= (E_n - \hbar\omega)\hat{a}_-|n >,$$

show that the lowest eigenvalue of $\hat{H}$ is $E_0 = \frac{1}{2}\hbar\omega$, that

$$E_n = (n + \frac{1}{2})\hbar\omega \qquad [5]$$

and that the eigenstate corresponding to E_n is

$$|n >= C_n(\hat{a}_+)^n|0 >,$$

where C_n is a normalisation constant. [2]

(e) Given that, in the x-representation, the eigenstate $|0 >$ is

$$\psi_0(x) = A_0 e^{-m\omega x^2/2\hbar},$$

show that

$$\psi_2(x) = A_2(1 - \frac{2m\omega x^2}{\hbar})e^{-m\omega x^2/2\hbar} \qquad [6]$$

where A_0 and A_2 are normalisation constants.

9. (a) A charged spin-1/2 particle is fixed in an external uniform magnetic field $\mathbf{B} = B\hat{z}$ where $\hat{z}$ is a unit vector in the z-direction. The Hamiltonian operator is

$$\hat{H} = \gamma B \hat{S}_z$$

where $\hat{S}_z$ is the z-component of the spin operator $\hat{\mathbf{S}}$ and γ is a constant. Find the energy eigenvalues of this system. [4]

(b) If $|\alpha>$ and $|\beta>$ are eigenvectors of $\hat{S}_z$ corresponding to eigenvalues $\hbar/2$ and $-\hbar/2$ respectively, determine ω such that the wave function at time t is

$$\Psi(t) = c_1 e^{-i\omega t/2}|\alpha> + c_2 e^{i\omega t/2}|\beta>,$$

where c_1 and c_2 are constants. [5]

(c) Suppose that at time $t = 0$ the system is in an eigenstate of $\hat{S}_x$,

$$\Psi(0) = \frac{1}{\sqrt{2}}(|\alpha> + |\beta>)$$

corresponding to eigenvalue $\hbar/2$. Write down the wave function at subsequent times and hence determine the times at which it will be an eigenstate of $\hat{S}_x$,

$$\Psi = \frac{1}{\sqrt{2}}(|\alpha> - |\beta>)$$

corresponding to eigenvalue $-\hbar/2$. [6]

(d) The operator $\hat{S}_n$ represents the component of the spin operator $\hat{\mathbf{S}}$ in the direction of a unit vector $\hat{n}$ lying in the $x - y$ plane:

$$\hat{S}_n = \hat{\mathbf{S}} \cdot \hat{n} = \hat{S}_x \cos\phi + \hat{S}_y \sin\phi$$

where ϕ is the angle between $\hat{n}$ and the x-axis.
Show that $\hat{S}_n$ may be expressed in matrix form as

$$\hat{S}_n = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$ [4]

(e) Show that the eigenvalues of $\hat{S}_n$ are $\pm\hbar/2$. [4]

(f) Determine the eigenfunction of $\hat{S}_n$ corresponding to the eigenvalue $\hbar/2$ and show that, if the system in initially in an eigenstate of $\hat{S}_x$, it will subsequently be in an eigenstate of $\hat{S}_n$ where $\hat{n}$ rotates in the $x - y$ plane with angular velocity ω. Comment on this result. [7]

10. (a) The Hamiltonian operator $\hat{H}$ describing a quantum mechanical system in spherical polar coordinates has a lowest energy eigenvalue E_0. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of $\hat{H}$ satisfies

$$E(F) = \frac{\int F^*(\mathbf{r})\hat{H}F(\mathbf{r})\mathrm{d}\mathbf{r}}{\int F^*(\mathbf{r})F(\mathbf{r})\mathrm{d}\mathbf{r}} \geq E_0$$ [8]

(b) Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit to its true value. [3]

(c) The Hamiltonian operator for a hydrogenic ion of nuclear charge number Z and reduced mass μ is, in atomic units,

$$\hat{H} = -\frac{1}{2\mu}\nabla^2 - \frac{Z}{r}.$$

Use the normalised three-dimensional trial function

$$F(\mathbf{r}) = \sqrt{\frac{\mu^3 Z^3}{\pi}}\,\mathrm{e}^{-\alpha r},$$

where α is a variational parameter, to derive an upper bound on the ground state energy of this system. [15]

(d) Demonstrate that this upper bound is also an exact energy for this system. [4]

END OF PAPER