UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Physics 3C26: Quantum Mechanics

COURSE CODE	:	**PHYS3C26**
UNIT VALUE	:	**0.50**
DATE	:	**05-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

05-C1093-3-130

Answer ALL questions from SECTION A and TWO from SECTION B.
The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

SECTION A.

1. Determine a real positive constant N such that the one-dimensional wave function
$$\psi(x) = Nxe^{-x^2/2} \quad , -\infty \leq x \leq \infty$$
is normalised to unity. [3]

   Give an expression for the expectation value of x^2 in the state $\psi(x)$, and calculate its value. [3]

   You may assume the standard integral for $n = 0, 1, 2 \cdots$ and $\alpha > 0$,
$$\int_{-\infty}^{+\infty} x^{2n} e^{-\alpha x^2} dx = \frac{\sqrt{\pi}(2n)!}{2^{2n} n! \alpha^{n+1/2}}$$

2. Two angular momentum operators $\mathbf{J_1}$ and $\mathbf{J_2}$ are coupled to form a resultant $\mathbf{J}$. If the corresponding quantum numbers are $j_1 = 1$ and $j_2 = 3$, what are the allowed values of j, the quantum number corresponding to $\mathbf{J}$? Write down the permitted values of the azimuthal quantum number m for each value of j. [6]

3. Define a **Hermitian operator.**

   Show that such an operator has real expectation values.

   Show also that if a system is in an eigenstate of an observable $\hat{A}$ the expectation value of the observable is equal to its eigenvalue if the state is normalised. [6]

4. Explain what is meant in quantum mechanics by (a) a superposition of eigenstates; (b) a pure state of an ensemble; (c) a mixed state of an ensemble.

   What is **decoherence?** How does decoherence explain the rare occurrence of superpositions of macroscopic systems (Schrödinger's cat states) in the physical world ? [8]

5. Define the **variance** $(\Delta\hat{X})^2$ of an observable $\hat{X}$ in a state Ψ. If $\hat{A}$ and $\hat{B}$ are two observables satisfying $[\hat{A}, \hat{B}] = i\hat{C}$ it may be shown that (but you are not required to do so)

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2} |< \hat{C} >| .$$

Discuss the significance of this inequality in quantum mechanics.

Derive its explicit form when $\hat{A} = \hat{x}$, the displacement operator and $\hat{B} = \hat{p}$, the x−component of linear momentum. [7]

6. What is meant in quantum mechanics by (a) **a hidden variable theory,** (b) **non-locality** ? [3]

State the criteria proposed by Einstein, Podolsky and Rosen for the basis of an acceptable quantum theory. [3]

What was the result when these criteria were put to an experimental test by Aspect and co-workers? [1]

SECTION B

7. Show that the component of spin $S_n = \mathbf{S}\cdot\hat{\mathbf{n}}$ of spin $\mathbf{S}$ of a spin-1/2 particle in a direction $\hat{n}$, where $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, is represented in the basis α, β where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by the matrix

$$S_n = \hbar/2 \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

[4]

[$\mathbf{S} = \frac{\hbar}{2}\sigma$ where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ and in the basis α, β the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.]$$

By solving the secular equation show that the eigenvalues of S_n are $\pm\hbar/2$ and that normalised eigenvector corresponding to eigenvalue $+\hbar/2$ is (to within an arbitrary phase factor)

$$| \chi_+^n >= \cos(\theta/2)\alpha + \sin(\theta/2)e^{i\phi}\beta$$ [12]

If the other normalised eigenvector is

$$| \chi_-^n >= -\sin(\theta/2)\alpha + \cos(\theta/2)e^{i\phi}\beta$$

then obtain α as a linear combination of $\chi_\pm^n$. [2]

A beam of electrons all with spin component $+\hbar/2$ along the z-axis moves along the x-axis. The beam passes through a Stern-Gerlach magnet whose magnetic field points in a direction $\hat{n}$ in the y-z plane at an angle $\theta = 50^0$ to the z-axis. How many beams emerge and what are their relative intensities?

[4]

If each emergent beam enters an ideal Stern-Gerlach filter which passes only electrons whose spin is in the +y direction, what is the probability of an electron emerging in each case ? [8]

8. A Hermitian operator A has a complete set of orthonormal eigenvectors $\mid n >$. Show that in the basis $\mid n >$, A is represented by a diagonal matrix with elements $A_{n'n} = a_n \delta_{n'n}$ where a_n is the eigenvalue of A corresponding to eigenvector $\mid n >$. [2]

If $\mathbf{J} = J_x\hat{i} + J_y\hat{j} + J_z\hat{k}$ is a quantum mechanical angular momentum operator, write down the commutation relation between J^2 and J_z. What does it tell us about these two observables? [3]

A particle has angular momentum operator $\mathbf{J}$. Its angular momentum quantum number $j = 3/2$. What are the eigenvalues of (a) J^2, (b) its z-component J_z? What are the matrices representing J^2 and J_z in the basis formed by the normalised eigenvectors of J^2 and J_z? [4]

Assume without proof that the normalised eigenvectors, labelled in descending order of corresponding eigenvalues of J_z are

$$\mid 1 >= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \mid 2 >= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \mid 3 >= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \mid 4 >= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Hamiltonian of the particle is $H = \frac{E_0}{\hbar^2}(J_x^2 + J_y^2) - \frac{5E_0}{2\hbar}J_z$.

Explain why the matrix representing H in this basis is diagonal. [4]

Determine this matrix. What are its eigenvalues? [6]

Assuming that the motion follows the time-dependent Schrödinger equation, write down the general form of the wave function $\psi(t)$ at time t. [3]

At $t = 0$ the system is described by the state vector

$$\psi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

What are the probabilities of finding, on measurement, each of the eigenvalues? [4]

The system is allowed to evolve with time. Find the time τ at which it first returns to its initial state. [4]

9. The Hamiltonian operator H describing a quantum mechanical system has a lowest energy eigenvalue E_0. Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of H satisfies

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0. \quad [8]$$

Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit on its value. [2]

Using the normalised trial function

$$F(x,a) = a^{-1/2} \cos \frac{\pi x}{2a}, \qquad -a \leq x \leq a,$$

$$F(x,a) = 0, \qquad |x| > a$$

where a is a variational parameter, show that for a particle of mass m in a one-dimensional harmonic oscillator potential of the form

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

$$E(F) = \frac{\pi^2 \hbar^2}{8ma^2} + \left(\frac{1}{6} - \frac{1}{\pi^2}\right) m\omega^2 a^2 \quad [12]$$

By minimising $E(F)$ with respect to variations in a, show that an upper bound on the ground state energy is

$$E = 0.568\hbar\omega. \quad [6]$$

Explain how this result is consistent with the requirements of the Variational Method. [2]

You may assume the result

$$\int x^2 \cos^2(\alpha x) dx = \frac{x^3}{6} + \frac{(2\alpha^2 x^2 - 1)}{8\alpha^3} \sin(2\alpha x) + \frac{x}{4\alpha^2} \cos(2\alpha x) + C$$

where α and C are constants.

10. The second-order perturbation theory formula may be written

$$W_n = E_n + \lambda V_{n,n} + \sum_{r \neq n} \frac{| \lambda V_{r,n} |^2}{E_n - E_r}.$$

(a) Explain carefully the meaning of each term in this relation and describe how it is used to calculate approximate energy levels to different orders in the perturbation. [7]

(b) Outline the steps involved in deriving this expression. (Precise mathematical detail is not required.) [7]

The eigenvalues E_n of a particle of mass m in an infinite square well potential

$$V(x) = \begin{cases} 0, & -a \leq x \leq a; \\ \infty, & | x | > a \end{cases}$$

are given by, for $n = 1, 2, 3 \cdots$,

$$E_n = \frac{\hbar^2 \pi^2 n^2}{8ma^2}.$$

The corresponding orthonormal eigenfunctions are the even and odd functions

$u_n = a^{-\frac{1}{2}} \cos(n\pi x/2a), \quad -a \leq x \leq a; \quad$ for $n = 1, 3, 5, \cdots$

$u_n = a^{-\frac{1}{2}} \sin(n\pi x/2a), \quad -a \leq x \leq a; \quad$ for $n = 2, 4, 6, \cdots$

$u_n = 0, \quad$ if $| x | > a$.

A particle moves in a potential given by

$$V(x) = \begin{cases} V_0(1 + \cos(\pi x/a)), & -a \leq x \leq a; \\ \infty, & | x | > a \end{cases}$$

where $V_0 << \frac{\pi^2 \hbar^2}{8ma^2}$. Treat this problem as a perturbation on the case of a particle in an infinite square well potential. Show that the first-order correction to the ground-state energy can be written as

$$\frac{V_0}{2} \left(\int_{-a}^{a} u_1(x) u_3(x) dx + 3 \int_{-a}^{a} u_1^2(x) dx \right)$$ [8]

(You may assume that $\cos(\frac{\pi x}{a}) \cos(\frac{\pi x}{2a}) = \frac{1}{2}\left(\cos(\frac{3\pi x}{2a}) + \cos(\frac{\pi x}{2a})\right)$.)

Show that the ground state energy to second order in V_0 is given by

$$\frac{\pi^2 \hbar^2}{8ma^2} + \frac{3}{2} V_0 - \frac{V_0^2 ma^2}{4\pi^2 \hbar^2}$$ [8]

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