

UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. M.Sci.

Physics 3C26: Quantum Mechanics

COURSE CODE : **PHYS3C26**

UNIT VALUE : **0.50**

DATE : **10-MAY-04**

TIME : **10.00**

TIME ALLOWED : **2 Hours 30 Minutes**

Answer ALL questions from SECTION A and TWO from SECTION B.

The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

SECTION A.

1. Determine a real positive constant N such that the wave function

$$\psi(\mathbf{r}, t) = Ne^{-r/2}(3\cos^2\theta - 1)e^{i\omega t}$$

given in spherical polar coordinates is normalised to unity in three-dimensional space. [4]

Calculate the expectation value of r^2 in the state $\psi(\mathbf{r}, t)$. [3]

(You may assume that if n is a positive integer $\int_0^\infty e^{-r}r^n dr = n!$.)

2. What is meant by the statement that two observables are **compatible**?

Show that if \hat{A} and \hat{B} are compatible, they commute. [6]

3. Define a **Hermitian operator**.

Show that the eigenvalues of such an operator are real and that the eigenfunctions corresponding to eigenvalues that are different are orthogonal. [7]

4. What is meant in quantum mechanics by the phrase **collapse of the wave function**? [2]

What is the **Copenhagen interpretation** of quantum mechanics? [2]

Explain, illustrating your explanation by an example, what is meant in quantum theory by **complementarity**. [3]

5. In the z-direction basis

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the component $S_n = \mathbf{S} \cdot \hat{\mathbf{n}}$ of the spin \mathbf{S} of a spin $\frac{1}{2}$ particle in a direction $\hat{\mathbf{n}}$ where $\hat{\mathbf{n}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is given by

$$S_n = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}$$

Show that the eigenvalues of S_n are $\pm \hbar/2$. [4]

Assume that the normalised eigenvector corresponding to eigenvalue $\hbar/2$ is

$$|\chi_+\rangle = \cos(\theta/2)\alpha + \sin(\theta/2)e^{i\phi}\beta.$$

A beam of electrons is spin polarised with component $+\hbar/2$ in a direction defined by the angles $\theta = \pi/3, \phi = \pi/4$.

What would be the probability of finding $+\hbar/2$ in a measurement of the component of spin in the positive z-direction? [1]

What would be the result of a second measurement of this component immediately after the first? Explain your answer. [2]

6. The Hamiltonian matrix of a quantum system is given by

$$\mathbf{H} = \begin{pmatrix} \lambda & -\lambda \\ -\lambda & 1 \end{pmatrix}$$

where λ is a real, positive constant with

$$\lambda \ll 1.$$

Decompose this Hamiltonian into unperturbed and perturbed parts in the form

$$\mathbf{H} = \mathbf{H}_0 + \lambda \mathbf{V}$$

and apply the second-order perturbation theory formula

$$W_n = E_n + \lambda V_{n,n} + \lambda^2 \sum_{m \neq n} \frac{|V_{m,n}|^2}{E_n - E_m}.$$

to determine the eigenvalues of \mathbf{H} to second-order in λ . [6]

SECTION B

7. If \mathbf{J} is a quantum mechanical angular momentum operator, write down the commutation relations

(a) among the Cartesian components of \mathbf{J} , (b) between J^2 and J_z . [2]

If J_+ and J_- are defined by $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$, show that

$$[J_z, J_+] = \hbar J_+ ; \quad [J_z, J_-] = -\hbar J_- , \quad [2]$$

Hence, if

$$J_z |j, m\rangle = m\hbar |j, m\rangle$$

show that $J_+ |j, m\rangle$ and $J_- |j, m\rangle$ are proportional to $|j, m+1\rangle$ and $|j, m-1\rangle$ respectively. [2]

In the following you may assume that this proportionality takes the form

$$J_{\pm} |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle .$$

For a spin- $\frac{1}{2}$ system, $\mathbf{J} = \mathbf{S}$ where

$$S^2 |s, m\rangle = s(s+1)\hbar^2 |s, m\rangle, \quad S_z |s, m\rangle = m\hbar |s, m\rangle$$

with $s = \frac{1}{2}$ and $m = \pm \frac{1}{2}$. Using the notation $|\frac{1}{2}, \frac{1}{2}\rangle = \alpha$, $|\frac{1}{2}, -\frac{1}{2}\rangle = \beta$ show that

$$S_+ \alpha = 0; \quad S_+ \beta = \hbar \alpha; \quad S_- \alpha = \hbar \beta; \quad S_- \beta = 0. \quad [4]$$

Hence show that the matrices representing S_x and S_y in the basis α, β are

$$\mathbf{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{S}_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad [4]$$

The Hamiltonian of a spin- $\frac{1}{2}$ system is

$$H = E_0(S_x^2 + S_y^2 - \hbar S_x)$$

where E_0 is a real, positive constant. Find the matrix of H and determine its eigenvalues and normalised eigenvectors. [6]

Write down the general form of the wave function $\psi(t)$ at time t . [3]

If $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ find the probability of finding on measurement each of the eigenvalues. [3]

Find also the times t at which the system returns to its initial state. [4]

8. State the **Principle of Superposition** in quantum mechanics. [2]

What are meant in quantum mechanics by

Entanglement ? [2]

Non-locality ? [2]

A qubit ? [2]

The four so-called Bell states of a two-photon system are defined to be

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle \pm |0\rangle|1\rangle)$$

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle|1\rangle \pm |0\rangle|0\rangle)$$

Alice wishes to teleport a photon A in an unknown state

$$|T\rangle = C_1|1\rangle + C_2|0\rangle$$

where C_1, C_2 are complex constants satisfying

$$|C_1|^2 + |C_2|^2 = 1$$

to a colleague Bob. A pair of ancillary photons (B,C) is prepared in the Bell state $|\Psi^-\rangle$. What is the resulting state vector of the three-photon system A,B and C? [5]

Regroup this state vector in terms of Bell states of the subsystem (A,B) and single-photon states of C. What is the probability of finding the pair (A,B) on measurement in any one given Bell state? [2]

Using these results, describe the procedure whereby Alice may teleport her photon to a colleague Bob in a different location. [10]

9. The Hamiltonian operator H describing a quantum mechanical system has a lowest energy eigenvalue E_0 . Show, for any normalisable function $F(\mathbf{r})$ that satisfies the boundary conditions appropriate to a bound state, that the expectation value $E(F)$ of H satisfies

$$E(F) = \frac{\int F(\mathbf{r})^* H F(\mathbf{r}) d\mathbf{r}}{\int F(\mathbf{r})^* F(\mathbf{r}) d\mathbf{r}} \geq E_0.$$

Explain how this expression can be used to find an approximation to the ground state energy which is an upper limit on its value. [8]

Use a trial function of the form

$$F(x, \alpha) = e^{-\alpha x^2/2}$$

where α is a variational parameter, to calculate a variational estimate of the ground state energy of a particle of mass m in a one-dimensional Harmonic oscillator potential of the form

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

By minimising the expectation value $E(\alpha)$ of H with respect to variations in α , show that an upper bound on the ground state energy is

$$E = \frac{1}{2}\hbar\omega. \quad [11]$$

Discuss how the variational method could be extended to find approximate energy eigenvalues for excited states. [4]

In the case of the potential discussed above, suggest an appropriate form for the trial function for:

(a) the first excited state. [3]

(b) the second excited state. [4]

You may assume the standard integral for $n = 0, 1, 2 \dots$ and $\alpha > 0$,

$$\int_{-\infty}^{+\infty} x^{2n} e^{-\alpha x^2} dx = \frac{\sqrt{\pi}(2n)!}{2^{2n} n! \alpha^{n+1/2}}$$

10. The creation and annihilation operators a_+ and a_- for a one-dimensional harmonic oscillator of mass m and angular frequency ω are defined by

$$a_+ = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega x), \quad a_- = a_+^\dagger,$$

where x and p are position and momentum operators satisfying $[x, p] = i\hbar$. From these definitions show that the commutator

$$[a_-, a_+] = 1. \quad [2]$$

Show further that if N is the operator defined by $N = a_+a_-$

$$[N, a_+] = a_+ \quad ; [N, a_-] = -a_-. \quad [3]$$

Using the notation $N | n \rangle = \lambda_n | n \rangle$ show that

$$Na_\pm | n \rangle = (\lambda_n \pm 1) | n \rangle \quad [3]$$

Show that the eigenvalues λ_n of N are $n = 0, 1, 2, \dots$ and give a physical interpretation of N . [4]

A state $|\alpha\rangle$ that is an eigenstate of the annihilation operator a_- , i.e. $a_- |\alpha\rangle = \alpha |\alpha\rangle$ with eigenvalue α is called a coherent state. Expand $|\alpha\rangle$ in terms of the complete set of eigenstates of the operator N as

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$$

and show that

$$C_1 = \alpha C_0; \quad C_2 = \frac{\alpha}{\sqrt{2}} C_1; \quad C_3 = \frac{\alpha}{\sqrt{3}} C_2; \dots C_n = \frac{\alpha}{\sqrt{n}} C_{n-1} \quad [6]$$

Hence show that the normalised state $|\alpha\rangle$ is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad [6]$$

Find the probability that the coherent state contains k quanta and also the average number of quanta in the coherent state. [6]

You should assume that

(i) For any two states $|\mu\rangle$ and $|\nu\rangle$, then $\langle \mu | a_-^\dagger \nu \rangle = \langle a_- \mu | \nu \rangle$.

(ii) $a_- |n\rangle = \sqrt{n} |n-1\rangle$.