UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

Physics 2B72: Mathematical Methods

COURSE CODE	:	**PHYS2B72**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

05-C1088-3-40

TURN OVER

All questions may be attempted. *Credit will be given for all work done correctly. Numbers in square brackets show the provisional allocation of marks per sub-section of the question.*

1. (a) The matrices $\underline{A}$, $\underline{B}$ and $\underline{D}$ are related by $\underline{D} = \underline{A}\underline{B}$.
Given that

$$\underline{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \underline{D} = \begin{pmatrix} 6 & 5 & 6 \\ 24 & 26 & 16 \\ 5 & 16 & -3 \end{pmatrix},$$

evaluate $\underline{A}^{-1}$ and use this result to find the matrix $\underline{B}$. [10 marks]

(b) If † denotes the Hermitian conjugation, show that

$$(\underline{A}\underline{B})^{\dagger} = \underline{B}^{\dagger}\underline{A}^{\dagger}.$$ [3 marks]

The trace of a matrix is defined as the sum of its diagonal elements,

$$\mathrm{Tr}\,\{\underline{C}\} = \sum_i C_{ii}.$$

By writing out the matrix multiplication explicitly in terms of components, show that for any matrix $\underline{S}$ the trace of $\underline{C} = \underline{S}^{\dagger}\underline{S}$ can never be negative. [3 marks]

Verify this result explicitly in the case where

$$\underline{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ [4 marks]

2. The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Verify that one of the eigenvalues is $\lambda_1 = 0$ and that the corresponding eigenvector is $\underline{v_1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. **[5 marks]**

Find the characteristic equation and the two other eigenvalues λ_2 and λ_3. **[4 marks]**

Find the normalized eigenvectors $\underline{v_2}$ and $\underline{v_3}$ corresponding to λ_2 and λ_3. **[8 marks]**

Show that the eigenvectors of $\underline{A}$ are orthogonal. **[3 marks]**

3. (a) Verify the following vector equations

$$\underline{\nabla}.(\underline{A} \times \underline{B}) = \underline{B}.(\underline{\nabla} \times \underline{A}) - \underline{A}.(\underline{\nabla} \times \underline{B})$$

for the vector functions

$$\begin{aligned} \underline{A} &= 3y^2x\underline{\hat{e}}_x + xy\underline{\hat{e}}_y + z^2\underline{\hat{e}}_z \\ \underline{B} &= 2x^3\underline{\hat{e}}_x + 4yz^2\underline{\hat{e}}_y + yx\underline{\hat{e}}_z. \end{aligned}$$

[10 marks]

(b) The function $u(x,t)$ satisfies the differential equation

$$\left(\frac{\partial^2 u}{\partial t^2}\right) + \alpha^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2}\right),$$

where c and α are real constants.

By seeking a solution of the equation in the separable form $u(x,t) = X(x) \times T(t)$, find the most general solution for which $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = 0$. **[10 marks]**

4. (a) Show that the second-order differential equation

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \qquad a_0 \neq 0$$

with $k = 0$ or $k = \frac{1}{2}$. **[6 marks]**

Derive the recurrence relation

$$\frac{a_{n+1}}{a_n} = \frac{-1}{2(n+k+1)(2n+2k+1)}\,.$$ **[4 marks]**

Show that for $k = \frac{1}{2}$ the explicit form for the solution for y as a function of x is

$$y(x) = \sqrt{x}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \ldots\right).$$ **[3 marks]**

Show that for $k = 0$ the explicit form for the solution for y as a function of x is

$$y(x) = 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \ldots.$$ **[3 marks]**

(b) The recurrence relation for another second-order differential equation is found to be

$$a_{n+1} = \frac{p^2 - n(n-1) - 3}{n+1}a_n.$$

Show, using the d'Alembert ratio test, that unless the parameter p is chosen to make the series terminate it will diverge.
Find the value of p for which the series has $a_n = 0$ for $n \geq 4$. **[4 marks]**

5. The generating function for the Legendre polynomials is

$$g(x,t) \equiv (1 - 2x\,t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n\,,$$

where $|t| \leq 1$.

(a) By expanding $g(x,t)$ in powers of t, show that

$$P_0(x) = 1,\ P_1(x) = x,\ \text{and}\ P_2(x) = \tfrac{1}{2}(3x^2 - 1).$$ **[3 marks]**

(b) By differentiating $g(x,t)$ with respect to x, show that the Legendre polynomials satisfy the recurrence relation

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x).$$ **[6 marks]**

(c) Use the recurrence relation to find the expression for $P'_3(x)$. **[3 marks]**

(d) The orthogonality and normalization of the Legendre polynomials is given by

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = \frac{2}{2m+1}\,\delta_{mn}.$$

Explain what is meant by the right hand side of the relation. **[3 marks]**

(e) By expressing the integrand of the following integral in terms of a sum of Legendre polynomials show that

$$\int_{-1}^{+1} \left[\frac{1}{2}\left(1 + \sqrt{3}x\right)^2 - \frac{1}{2}\right]^2 dx = 2.9$$

[5 mark]

6. If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, ,$$

show, by using the orthonormality of the sine and cosine functions, that the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \, , \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \, . \end{aligned}$$

[7 marks]

The function $f(x)$ is periodic with period 2π. In the interval $-\pi < x < +\pi$, it is given by

$$f(x) = x + \pi$$

Sketch $f(x)$ and show that the function can be expressed as a sum of an even function and an odd function. **[2 marks]**

Show that the Fourier series of this function is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \ldots\right) .$$

[8 marks]

Discuss the behaviour of $f(x)$ at $x = 0$ and as x tends to π using both the explicit function and the Fourier series. **[3 marks]**

7. State Stokes' theorem in integral form **[2 marks]**

Calculate the line integral $I = \oint_\gamma \underline{W} \cdot \underline{d\ell}$ of the vector

$$\underline{W} = -xy^2\hat{\underline{e}}_x + xy\hat{\underline{e}}_y + xy\hat{\underline{e}}_z,$$

where the closed contour γ is the perimeter of the square with vertices at (0,0,0), (1,0,0),(1,1,0) and (0,1,0) in that order. **[7 marks]**

Verify Stokes' theorem for the vector $\underline{W}$ over the surface of a cube with edges of unit length bounded by the contour γ and with $z > 0$. **[7 marks]**

Explain without integration why

$$\int_{S_0} (\underline{\nabla} \times \underline{W}) \cdot \hat{\underline{n}} dS_0 = 1$$

where S_0 is the square with vertices (0,0,0), (1,0,0), (1,1,0) and (0,1,0), $\hat{\underline{n}}$ $=+\hat{\underline{e}}_z$ and $\underline{W}$ is the vector given above. **[4 marks]**

END OF PAPER