UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

Physics 2B72: Mathematical Methods

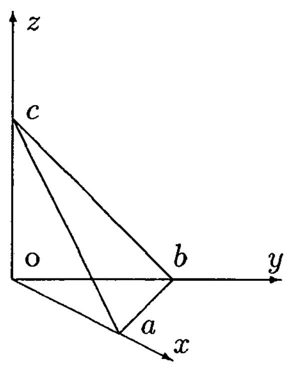
COURSE CODE	:	**PHYS2B72**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

04-C1089-3-40

TURN OVER

All questions may be attempted. *Credit will be given for all work done correctly. Numbers in square brackets show the provisional allocation of marks per sub-section of the question.*

1. State the divergence theorem. **[2 marks]**



The tetrahedron shown in the picture has vertices placed at $a = (1,0,0)$, $b = (0,1,0)$, $c = (0,0,1)$.

Show that the area of the triangle abc is equal to $\frac{1}{2}\sqrt{3}$. **[2 marks]**

Find the normal $\underline{n}$ to the plane abc and show that the equation of the plane is $\underline{n} \cdot \underline{r} = x + y + z = 1$. **[2 marks]**

Evaluate the volume of the tetrahedron

$$V = \int_V dV = \int_0^1 dz \int_0^{1-z} dy \int_0^{1-z-y} dx \; .$$ **[2 marks]**

Verify the divergence theorem for the tetrahedron for the vector field $\underline{F} = \underline{r}$. **[5 marks]**

Integrate the flux of the vector field

$$\underline{F} = y^2 \underline{\hat{e}}_x + z^2 \underline{\hat{e}}_z$$

over the three faces oab, obc, and oca. **[3 marks]**

Also by integrating $\underline{\nabla} \cdot \underline{F}$ over the volume of the tetrahedron, use the divergence theorem to deduce the flux of $\underline{F}$ through the slanted surface abc. **[4 marks]**

2. (a) By writing both sides of the equation explicitly in Cartesian coordinates, prove the identity

$$\underline{\nabla}\,(\underline{A}\cdot\underline{B}) = (\underline{A}\cdot\underline{\nabla})\;\underline{B} + (\underline{B}\cdot\underline{\nabla})\;\underline{A} + \underline{A}\times(\underline{\nabla}\times\underline{B}) + \underline{B}\times(\underline{\nabla}\times\underline{A})\ ,$$

where $\underline{A}$ and $\underline{B}$ are vector functions of x, y, and z. **[8 marks]**

(b) The function $u(x,t)$ satisfies the differential equation

$$\left(\frac{\partial^2 u}{\partial t^2}\right) - \alpha^2\, u = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)\ ,$$

where c and α are real positive constants.

By seeking a solution of the equation in the separable form $u(x,t) = X(x)\times T(t)$, find the most general solution for which $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,t)\to 0$ as $t\to\infty$. **[10 marks]**

What is the minimum value of α for which a solution exists? **[2 marks]**

3. (a) The matrices $\underline{A}$, $\underline{B}$, and $\underline{D}$ are related by $\underline{D} = \underline{A}\,\underline{B}$. Given that

$$\underline{A} = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix} \quad \text{and} \quad \underline{D} = \begin{pmatrix} 10 & -7 & -3 \\ -3 & 8 & -2 \\ 3 & -14 & 7 \end{pmatrix},$$

evaluate $\underline{A}^{-1}$. **[7 marks]**

Hence derive the value of $\underline{B}$. **[3 marks]**

(b) For the matrices

$$\underline{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \underline{C} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

calculate $\underline{B}^2$ and $\underline{B}^3$ and show that, for non-negative integers n,

$$\underline{B}^{2n+1} = 2^n\,\underline{B}\,,$$
$$\underline{B}^{2n+2} = 2^n\,\underline{C}\,.$$

[6 marks]

By expanding the exponential in a power series in α, show that

$$\exp(\alpha\underline{B}) = \underline{I} - \tfrac{1}{2}\underline{C} + \tfrac{1}{2}\cosh\left(\alpha\sqrt{2}\right)\underline{C} + \tfrac{1}{\sqrt{2}}\sinh\left(\alpha\sqrt{2}\right)\underline{B}\,,$$ **[4 marks]**

where $\cosh x = \sum_{n=0}^{\infty} x^{2n}/(2n)!$

4. The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{pmatrix} -3 & 2i & 2 \\ -2i & 1 & -3i \\ 2 & 3i & 1 \end{pmatrix}.$$

Verify that one of the eigenvalues is $\lambda_1 = -2$ and that the corresponding *normalised* eigenvector is $\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$. **[5 marks]**

By showing that the characteristic equation is $\lambda^3 + \lambda^2 - 22\lambda - 40 = 0$, or otherwise, find the other two eigenvalues λ_2 and λ_3 and the associated normalised eigenvectors $\underline{v}_2$ and $\underline{v}_3$. **[9 marks]**

Show explicitly that these eigenvectors are mutually orthogonal, $\underline{v}_i^\dagger\underline{v}_j = 0$ for $i \neq j$. **[3 marks]**

Why should this be so? **[1 mark]**

Show further that

$$\underline{v}_1 \times \underline{v}_2 = C\,\underline{v}_3^*\,,$$

where the constant C has magnitude one. **[2 marks]**

5. Show that the second-order differential equation

$$(2x - 2x^2)\,\frac{d^2y}{dx^2} + (1 - x)\,\frac{dy}{dx} + 3y = 0$$

has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \qquad a_0 \neq 0$$

with $k = 0$ or $k = \frac{1}{2}$. **[6 marks]**

Derive the recurrence relation

$$\frac{a_{n+1}}{a_n} = \frac{(n+k)(2n+2k-1) - 3}{(n+k+1)(2n+2k+1)}\,.$$ **[4 marks]**

Show that the $k = \frac{1}{2}$ series terminates and find the explicit form for the solution for y as a function of x. **[3 marks]**

Use the d'Alembert ratio test to determine the range of values of x for which the $k = 0$ series converges. **[3 marks]**

Explain why, from the structure of the differential equation, one would expect the solution to <u>either</u> vanish at $x = 1$ <u>or</u> to have a badly behaved solution at $x = 1$. **[4 marks]**

6. If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx ,$$

show, by quoting the orthonormality of the sine and cosine functions, that the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx , \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx . \end{aligned}$$

[6 marks]

The function $f(x)$ is periodic with period 2π. In the interval $-\pi < x < +\pi$, it is given by

$$f(x) = \begin{cases} \sin x & \text{if } x > 0 , \\ -\sin x & \text{if } x < 0 . \end{cases}$$

Is $f(x)$ even or odd? **[1 mark]**

Show that the Fourier series of this function is

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{\substack{n \text{ even} \\ n \geq 2}}^{\infty} \frac{1}{n^2 - 1} \cos nx .$$

[8 marks]

Hence write down the Fourier series for the periodic function given by $g(x) = \cos x$ for $0 < x < \pi$ and $g(x) = -\cos x$ for $-\pi < x < 0$. **[3 marks]**

Use the Fourier series for $f(x)$ at $x = 0$ to evaluate the sum

$$S = \sum_{\substack{n \text{ even} \\ n \geq 2}}^{\infty} \frac{1}{n^2 - 1} .$$

Verify the order of magnitude of your answer by evaluating the sum of the first five terms on a calculator. **[2 marks]**

You may find the following identity useful:

$$\sin A \cos B = \tfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right] .$$

7. The generating function for the Legendre polynomials is

$$g(x,t) \equiv (1 - 2x\,t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n \,,$$

where $|t| \leq 1$.

(a) Show that $P_n(1) = 1$. **[2 marks]**

(b) Show that $P_n(x) = (-1)^n P_n(-x)$. **[2 marks]**

(c) By expanding $g(x,t)$ in powers of t, show that

$$P_0(x) = 1,\ P_1(x) = x, \text{ and } P_2(x) = \tfrac{1}{2}(3x^2 - 1).$$ **[3 marks]**

(d) By differentiating $g(x,t)$ with respect to t, show that the Legendre polynomials satisfy the recurrence relation

$$(n+1)\,P_{n+1}(x) - (2n+1)\,x\,P_n(x) + n\,P_{n-1}(x) = 0 \,.$$ **[5 marks]**

(e) Use the recurrence relation to find the expression for $P_3(x)$. **[1 mark]**

(f) Find the values of x satisfying $P_2(x) = 0$ and those satisfying $P_3(x) = 0$. **[2 marks]**

(g) Why does orthogonality of the Legendre polynomials require that the solutions for x in part (f) lie in the range $-1 < x < +1$? **[2 marks]**

(h) For $x \gg 1$ the leading term in the Legendre polynomial is

$$P_n(x) = \alpha_n\,x^n \,.$$

Use the recurrence relation to show that

$$\alpha_n = \frac{(2n-1)!!}{n!} \,,$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots\cdots 1$ for $n \geq 1$. **[3 marks]**

END OF PAPER