UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Physics 2B28: Statistical Thermodynamics and Condensed Matter Physics

COURSE CODE	:	**PHYS2B28**
UNIT VALUE	:	**0.50**
DATE	:	**11-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

05-C1085-3-130

TURN OVER

Answer ALL questions from section A and THREE questions from section B

The numbers in square brackets in the right hand margin indicate the provisional allocation of marks per sub-section of a question.

Boltzmann's constant $k = 1.38 \times 10^{-23}$ J K^{-1}
Planck's constant $h = 6.627 \times 10^{-34}$ J s
Electronic charge $e = 1.6 \times 10^{-19}$ J
Avogadro's number $N_A = 6 \times 10^{23}$
Mass of electron $m_e = 9.1 \times 10^{-31}$ kg
$\beta = (kT)^{-1}$

SECTION A

1. (a) What is meant by the terms *macrostate, microstate* and *statistical weight* in statistical thermodynamics? [2]

(b) Write down Boltzmann's expression for the entropy of an isolated system, explaining the meaning of each symbol in the expression. [1]

(c) An isolated system of 6 magnetic dipoles is in a macrostate with 4 dipoles 'up' and 2 dipoles 'down'. Calculate the statistical weight of this macrostate. [2]

2. (a) Write down the Kelvin *and* Clausius statements of the second law of thermodynamics [2]

(b) State the third law of thermodynamics, and discuss the possibility of reducing the temperature of a system to absolute zero. [4]

3. (a) State the *Boltzmann distribution* of a system in equilibrium at a temperature T, explaining the meaning of each term. [3]

(b) Show that the average energy of a system at equilibrium at a temperature T can be expressed as $-(\partial \ln Z / \partial\beta)$ where the symbols have their usual meaning. [4]

4. (a) What are the possible spin values of identical particles which obey Fermi-Dirac statistics (i.e. fermions)? Give 2 examples of such particles. [2]

(b) The *Fermi-Dirac occupation function* is given by

$$n(E) = \{\exp [(E-E_F)/kT] + 1\}^{-1}$$

How are the *Fermi energy* E_F and *Fermi temperature* T_F defined? [2]

Draw $n(E)$ as a function of E, at $T = 0$, and at a temperature $T << T_F$, marking E_F clearly on your diagram. [3]

5. Two *identical* particles can occupy three single-particle states with energies $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon$ and $\varepsilon_3 = 2\varepsilon$.

(a) If the two particles are *fermions*, draw all of the possible arrangements in which they can occupy the single-particle states. State the total energy of the two particles in each case, and hence write down the partition function. [4]

(b) Draw the additional ways in which two *bosons* can occupy the single particle states, giving their total energies. What is the partition function in this case? [3]

6. Explain briefly what is meant by the *free-electron model* of a metal. [2]

Show that the density of states of a free electron gas, as a function of wave vector k, is given by $f(k)\, dk = V k^2\, dk/(2\pi^2)$. [5]

How is $f(k)$ modified when the effect of electron spin is included? [1]

SECTION B

7. (a) A particular atom can exist in three energy states: $E_1 = 1.1 \times 10^{-22}$ J, $E_2 = 1.9 \times 10^{-22}$ J and $E_3 = 3.2 \times 10^{-22}$ J, with degeneracies $g_1 = 1$, $g_2 = 3$ and $g_3 = 5$, respectively. If the atom is in equilibrium at a temperature of 10 K, calculate:

(i) the value of the partition function Z [2]

(ii) the individual probabilities that the atom has energy $E_i = E_1, E_2$ and E_3 [3]

(iii) the mean energy per atom [2]

(b) Now use the expression $S = -k \sum p_i \ln p_i$ to calculate the entropy per atom [5]

(c) The energy levels of a simple spin $^1/_2$ magnetic system in a magnetic field B are $E_1 = -\Delta$ and $E_2 = +\Delta$, where $\Delta = \mu_B B$. Using the expression in part (b) show that the entropy of the system is

$$S = 2\,\Delta\,\{T\,[\,1 + \exp\,(2\,\Delta/kT)]\}^{-1} + k\;\ln\,[1 + \exp\,(-2\,\Delta/kT)]$$

[6]

(d) By considering this formula, deduce the entropy of the system in both the low temperature and high temperature limits. [2]

8. (a) State Clausius' entropy formulation of the Second Law of Thermodynamics? What condition does the entropy of an isolated system satisfy when it is in thermodynamic equilibrium? [2]

(b) An isolated system is partitioned into two sub-systems, 1 and 2. By considering the change in entropy as heat flows from sub-system 1 to sub-system 2, derive the condition that must be satisfied for sub-systems 1 and 2 to be in thermal equilibrium. Show that this leads to the definition of temperature:

$$1/T = (\partial S / \partial E)_{N,V} \quad [4]$$

(c) If the two sub-systems discussed in part(b) are *not* in thermal equilibrium, show that heat flows from the sub-system at higher temperature to the sub-system at lower temperature. [4]

(d) A linear chain of N+1 spins forms a one-dimensional paramagnetic system. Each spin interacts with its neighbours in such a way that the total domain wall energy of the system is $E = n\varepsilon$, where n is the number of domain walls separating regions of down spins from up spins, and ε is the energy of one domain wall. These domain walls are indicated by vertical lines in the diagram below:

↓↓↓↓↓↓↓ | ↑↑↑↑ | ↓↓↓↓↓↓↓ | ↑↑↑↑↑↑↑↑ | ↓↓↓↓↓↓↓↓ | ↑↑↑↑↑ | ↓↓↓↓↓↓

How many ways can the n domain walls be arranged in the chain of $N+1$ spins? [2]
Calculate the entropy $S(E)$, and hence show that the energy E at a temperature T is [8]

$$E = N\varepsilon \{ [\exp (\varepsilon / k T)] + 1 \}^{-1}$$

[note Stirling's formula for large N: $\ln (N!) \sim N \ln (N) - N$]

9. (a) State the conditions when an ideal gas of identical quantum particles may be described as a *classical gas*, or as being in the *classical regime*. [2]

(b) Considering only the translational motion, show that in the classical regime the partition function of a single gas atom of mass m in a volume V is given by [6]

$$Z(1, V, T) = V \left(\frac{2\pi mkT}{h^2} \right)^{3/2}$$

You may assume that the density of momentum states is given by

$f(p)\, dp = (V / h^3)\, 4\, \pi p^2\, dp$ and that $\int_0^\infty x^2 \exp(-ax^2)\, dx = (1/4a)\, (\pi / a)^{1/2}$

(c) Show that the validity of the classical regime can be expressed as [4]

$$\frac{N}{V} \left(\frac{h^2}{2\pi mkT} \right)^{3/2} \ll 1$$

(d) The mass of a ^{4}He atom is 6.67 x 10^{-27} kg. Consider whether the classical regime is valid for (i) liquid helium at 4.2 K, for which $N/V \sim 2 \times 10^{28}$ m^{-3} and (ii) gaseous helium at 273 K, for which $N/V \sim 1 \times 10^{26}$ m^{-3} [4]

(e) Define the validity of the classical regime in terms of the *de Broglie wavelength* of the particles. Use this definition to confirm your conclusion in (d) for the validity of the classical regime for gaseous helium at 273K. [4]

10 (a) Consider an ideal Fermi-Dirac gas of N electrons of mass m in a volume V. Using the expression for the density of states as a function of wave vector k, $f(k)\, dk = V k^2\, dk/(2\pi^2)$, express the density of states as a function of energy, and hence show that the Fermi energy ε_F is given by : [8]

$$\epsilon_F = \frac{h^2}{2m_e}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

(b) Show that the internal energy of a Fermi-Dirac gas at T=0 is given $(3/5)\, N\varepsilon_F$. [3]

(c) Gold (Au) is a monovalent metal, *i.e.* it has one conduction electron per atom. The atomic mass of Au is 196.97 and its density is 19300 kg m^{-3}.

Calculate the Fermi energy (in eV), the Fermi temperature, *and* the Fermi velocity for the conduction electrons in gold: [6]

Comment on the ratio of the Fermi velocity to the velocity of light [1]

Calculate the internal energy due to the conduction electrons in 1 gram of Au at T=0. [2]

11.(a) Define the Gibbs function G of a system. Write down the differential form (dG). [2]

(b) Deduce the condition for two phases of a homogeneous one-component system to be in equilibrium at constant pressure and temperature. [3]

(c) Derive the Clausius-Clapeyron equation

$$\frac{dp}{dT} = \frac{L}{T\Delta V}$$

relating to equilibrium between phases subject to a first-order phase transition, where L is the latent heat and ΔV is the volume change in passing from one phase to the other. [8]

(d) Sketch the p, T phase diagram applicable to most simple substances. Label the solid, liquid and vapour phases, and mark clearly the triple and critical points. How does the phase diagram for H_2O differ from this diagram? [4]

(e) When lead is melted at atmospheric pressure, the melting point is 327.5°C, the density decreases from 11010 to 10650 kg m^{-3}, and the latent heat is 24.5 kJ kg^{-1}. What is the melting point of lead at a pressure of 100 atmospheres? Assume atmospheric pressure is 10^5 Pa. [3]