UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

Physics 1B71: Mathematics

COURSE CODE	:	**PHYS1B71**
UNIT VALUE	:	**1.00**
DATE	:	**18-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

06-C1162-3-40

TURN OVER

All questions may be attempted.
Full marks will be given for correct answers to the equivalent of about five questions.
The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks per sub-section of a question.
A collection of useful formulae is attached after the end of the paper.

1. **(a)** The points A, B and C have position vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ given by

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}\,; \qquad \mathbf{v} = 5\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}\,; \qquad \mathbf{w} = -\mathbf{i} + 2\mathbf{j} + t\mathbf{k}\,.$$

Determine the two values of the parameter t for which there is a right angle between the sides of the triangle ABC that meet at the point C. **[5]**

(b) Obtain the vector equation of the plane for which $\mathbf{r}_0$ is the position vector of a known point on the plane and vector $\mathbf{n}$ is directed along the normal. **[3]**

Two planes are specified by the equations

$$2x - 3y + z = 17\,; \qquad x + y - 2z = 1\,.$$

Show that the point with position vector $\mathbf{r}_0 = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ lies on both planes and for each plane write down a vector that is perpendicular to that plane. **[2]**
Hence by considering the vector product of these two vectors obtain the equation of the line of intersection of the two planes. **[6]**

(c) The time-dependent position vector $\mathbf{r}(t)$ of a moving particle is given by

$$\mathbf{r} = v_0\,[\,(1 - t + 2t^2)\,\mathbf{i} + 2t\,\mathbf{j} + (3 + 3t - t^2)\,\mathbf{k}\,]\,,$$

The vectors $\mathbf{i}$ and $\mathbf{j}$ are in the horizontal plane, $\mathbf{k}$ is directed vertically upwards and v_0 is a constant speed. Determine the time $t = t_m$ at which the particle reaches its maximum height. What is its speed at time t_m? **[4]**

2. **(a)** State De Moivre's Theorem. **[2]**

Hence or otherwise prove that

$$\cos(6\theta) = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

$$\sin(6\theta) = 2\cos\theta\sin\theta\,(16\sin^4\theta - 16\sin^2\theta + 3)\,.$$

[7]

(b) Simplify

$$z = \frac{21 - 3\sqrt{3}\mathrm{i}}{1 - 2\sqrt{3}\mathrm{i}}$$

[4]

and express z^6 in the form $x + \mathrm{i}y$. Find all the distinct roots of $z^{\frac{1}{6}}$ and sketch them on an Argand diagram. **[7]**

3. **(a)** State Taylor's theorem for the expansion of a function $f(x)$ as a power series in $(x-a)$ about some fixed point $x=a$. **[2]**

By using Taylor's theorem for the expansion of the functions $f(x)$ and $g(x)$, obtain l'Hôpital's rule for evaluating a limit of the form

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$$

stating the conditions for which it can be used. **[4]**

(b) Hence or otherwise determine

$$\lim_{x\to 0}\left(\frac{x^3-\sin(2x^2)}{x\ln(1+x)}\right).$$ **[5]**

(c) Apply Taylor's theorem to obtain the expansion of the function

$$f(x)=\tan^{-1}(x)$$

about the point $a=4/5$ retaining the first three terms. **[5]**

Hence obtain two approximate values for $f(1)$ by retaining (i) the first two terms only, and (ii) the first three terms in this Taylor series. Calculate the exact result and obtain the percentage errors for these two approximate values. **[4]**

4. **(a)** Express the following set of simultaneous linear equations

$$2x+y+3z=1\,;\quad x-3y+5z=-3\,;\quad 3x+4y-z=10$$

in the form of a matrix equation

$$\mathbf{Ax}=\mathbf{b}\,.$$ **[1]**

Solve the equations for x, y and z, *either* by using the determinant method *or* by obtaining the inverse matrix $\mathbf{A}^{-1}$. **[8]**

(b) State Simpson's rule for the numerical evaluation of the integral

$$I=\int_a^b f(x)\,\mathrm{d}x\,,$$

where the function $f(x)$ is tabulated at equal intervals of width h for the range $a\le x\le b$. **[2]**

If

$$f(x)=x^2\exp(-x)\,,$$

tabulate $f(x)$ as a function of x for the range $0\le x\le 2$ at intervals of $h=0.25$. Hence obtain approximate values of the integral

$$I=\int_0^2 x^2\exp(-x)\,\mathrm{d}x$$ **[5]**

correct to five decimal places for the two choices of interval length $h=0.5$ and $h=0.25$.

Prove, by direct integration, that the exact result is $I=2-10\exp(-2)$ and comment on the relative accuracy of the two approximate values obtained. **[4]**

5. **(a)** Find the gradient vectors $\nabla\phi$ and $\nabla\psi$ of the functions

$$\phi = x^3y^2 + z^2x - zy + z\,; \qquad \psi = x^2y + z^3x$$

at the point (1,–1,1). **[4]**

Hence obtain a vector **n** that is perpendicular to both $\nabla\phi$ and $\nabla\psi$. **[3]**

If the vector **u** is defined by

$$\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k},$$

find the magnitude of its component in the direction of **n**. **[3]**

(b) Find the stationary points of the function

$$f(x,y) = x^3 + 2y^2 - 4xy - 4y - x^2 + 4x\,.$$ **[5]**

For each point determine whether it is a maximum, a minimum or a saddle point. **[5]**

6. **(a)** The function $y(x)$ is a solution of the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y = Q(x)\,,$$

where $P(x)$ and $Q(x)$ are known functions of x. Show that, by multiplying the equation through by the integrating factor

$$\exp\left(\int P(x)\,\mathrm{d}x\right),$$

the left-hand side of the equation can be written in terms of an exact differential. **[3]**

Hence obtain the general solution of the equation

$$(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+2)^3\,.$$ **[6]**

Show that if $y = 12$ at $x = 4$,

$$y = (x+2)[\tfrac{1}{2}(x+2)^2 - 16].$$ **[3]**

(b) Obtain the general solution of the second-order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0\,.$$ **[5]**

Hence find the solution of the equation for which $y = 1$ and $\mathrm{d}y/\mathrm{d}x = 2$ at $x = 0$. **[3]**

7. **(a)** The vector function $\mathbf{F}(x, y)$ is given by

$$\mathbf{F} = (5xy^2 + 3e^{-x})\,\mathbf{i} + (5x^2y - y^2)\,\mathbf{j}.$$

Evaluate the line integral

$$\int \mathbf{F.ds}$$

along the path between the points $(x, y) = (0,0)$ and $(2,4)$ defined by $y = x^2$. **[6]**

Why will the result be the same for any other path between the points (0,0) and (2,4)? **[4]**

(b) Polar coordinates r and θ are related to Cartesian coordinates x and y for a point in a plane by

$$x = r\cos\theta\,; \qquad y = r\sin\theta\,.$$

Sketch a diagram to justify geometrically that an element of area dA in polar coordinates is given by

$$\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta\,.$$ **[2]**

Prove that the surface integral

$$\int\int_S (2x^4 + y^2)\mathrm{d}x\mathrm{d}y = \frac{\pi a^4}{4}(a^2 + 1),$$ **[8]**

where the surface S in the (x-y) plane is bounded by the circle $x^2 + y^2 = a^2$.

8. **(a)** $Z(x, y)$ is a general differentiable function of the independent variables x and y, and u and v are another pair of independent variables such that

$$x = \exp(u + v)\,; \qquad y = \exp(u - v)\,.$$

Prove that

$$\frac{\partial Z}{\partial u} = x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y}; \qquad \frac{\partial Z}{\partial v} = x\frac{\partial Z}{\partial x} - y\frac{\partial Z}{\partial y}\,.$$ **[5]**

Hence show that

$$4xy\frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial^2 Z}{\partial u^2} - \frac{\partial^2 Z}{\partial v^2}.$$ **[5]**

(b) State Gauss' divergence theorem for the flux of a vector $\mathbf{F}$ across a surface S that encloses a volume V. **[2]**

The vector function $\mathbf{F}(x, y, z)$ is given by

$$\mathbf{F} = (x^3 - x^2)y\,\mathbf{i} + (y^3 - 2y^2 + y)x\,\mathbf{j} + (z^2 - 1)\,\mathbf{k}.$$

If the surface S is the unit cube with vertices at the points (0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1) and (0,1,1), verify that the divergence theorem is satisfied for this vector. **[8]**

END OF QUESTIONS

CONTINUE FOR FORMULA SHEET

1B71 Formula Sheet

Trigonometry

$$\sin(\alpha \pm \beta) = \sin\alpha \,\cos\beta \,\pm\, \cos\alpha \,\sin\beta\,, \qquad \cos(\alpha \pm \beta) = \cos\alpha \,\cos\beta \,\mp\, \sin\alpha \,\sin\beta\,,$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \quad \sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right),$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \quad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Trigonometric and Hyperbolic Functions

$$e^{x} = \cosh x + \sinh x\,, \qquad e^{i\alpha} = \cos\alpha + i\,\sin\alpha\,,$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}\,, \qquad \sinh x = \frac{e^{x}-e^{-x}}{2}\,,$$

$$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}\,, \qquad \sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}\,.$$

Calculus

(C is an integration constant and all angles are in radians)

$$\frac{d}{dx}\sin x = \cos x\,, \qquad \frac{d}{dx}\cos x = -\,\sin\,x\,,$$

$$\frac{d}{dx}\sinh x = \cosh x\,, \qquad \frac{d}{dx}\cosh x = \sinh\,x\,,$$

$$\frac{d}{dx}\ln\,x = \frac{1}{x}\,, \qquad \frac{d}{dx}e^{x} = e^{x}\,,$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C\,, \qquad \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C\,, \qquad (a>0)\,,$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C\,, \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\,\tan^{-1}\left(\frac{x}{a}\right) + C\,,$$

$$\int \tan x\,dx = -\ln(\cos x) + C\,, \qquad \int \sec x\,dx = \ln(\sec x + \tan x) + C\,.$$

Series

$$\sum_{n=0}^{N} n = \frac{1}{2} N(N+1)\,, \qquad \sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}\,,$$

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{1.2}\,x^2 + \frac{n(n-1)(n-2)}{1.2.3}\,x^3 + \ldots + \binom{n}{m}\,x^m + \ldots$$

If n is a positive integer then this is a finite series. Otherwise it is an infinite series which only converges for $|x| < 1$.

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n+1}\,\frac{x^n}{n} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{x^n}{n} \qquad (-1 \;< x \le +1)\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\,.$$

Vectors

If $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, and similarly for the vector $\mathbf{B}$, then the scalar (dot) product is given by

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z = A\,B\,\cos\theta\,,$$

where A and B are the magnitudes of the two vectors and θ is the angle between them.

The vector (cross) product can be expressed as a three–by–three determinant

$$\begin{aligned} \mathbf{A}\times\mathbf{B} &= (A_yB_z - A_zB_y)\,\mathbf{i} - (A_xB_z - A_zB_x)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\,. \end{aligned}$$

The magnitude of the vector product is given by

$$|\mathbf{A}\times\mathbf{B}| = AB\,|\sin\theta|\,.$$

END OF PAPER