UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Phys & Astro 2246: Mathematical Methods III

COURSE CODE	:	**PHAS2246**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

06-C1109-3-160

TURN OVER

Answer FIVE questions only.

Numbers in square brackets show the provisional allocation of marks per sub-section of the question.

1. (a) Solve the simultaneous equations $y_1 = 6x_1 + 5x_2$ and $y_2 = 5x_1 + 4x_2$ for x_1 and x_2. Putting both sets of equations in matrix form,

$$\underline{y} = \underline{C}\,\underline{x} \quad \text{and} \quad \underline{x} = \underline{D}\,\underline{y}\,,$$

write down the 2×2 matrices $\underline{C}$ and $\underline{D}$. Show explicity that $\underline{C}\,\underline{D} = \underline{I}$, where $\underline{I}$ is the 2×2 unit matrix. **[6 marks]**

(b) Evaluate the 4×4 determinant

$$\Delta = \begin{vmatrix} 1 & 3 & 0 & 3 \\ 3 & 2 & 4 & 1 \\ 2 & 3 & 2 & 1 \\ -2 & 3 & 0 & 3 \end{vmatrix} .$$ **[5 marks]**

(c) Define the terms **Hermitian matrix**, and **trace of a matrix.** **[2 marks]**

If $\underline{A}$ and $\underline{B}$ are non-singular square matrices and $\underline{I}$ the unit matrix, all of order n, and † denotes Hermitian conjugation, show that

$$(\underline{A}\,\underline{B})^\dagger = \underline{B}^\dagger \underline{A}^\dagger$$ **[2 marks]**

$$\text{trace}(\underline{A}\,\underline{B}) = \text{trace}(\underline{B}\,\underline{A})$$ **[2 marks]**

If $\underline{A}^3 = \underline{I}$ and $\underline{B}\,\underline{A} = \underline{A}^2\underline{B}$, prove that $\underline{A}\,\underline{B} = \underline{B}\,\underline{A}^2$. **[3 marks]**

2. A matrix $\underline{A}$ is said to be real and symmetric. What does this mean? Prove that the eigenvalues of a real symmetric matrix are real. **[5 marks]**

A matrix is given by

$$\underline{A} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Obtain the characteristic equation satisfied by the eigenvalues of this matrix. Show that one eigenvalue is $\lambda = 2$ and hence obtain the other eigenvalues for $\underline{A}$. **[7 marks]**

Show that the normalised eigenvector associated with $\lambda = 2$ can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

Find the other two eigenvectors normalised to unit length. **[6 marks]**

For one pair of eigenvectors, demonstrate that they are orthogonal. **[2 marks]**

3. The generating function for the Legendre polynomials is

$$g(x,t) \equiv (1 - 2x\,t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n\,, \qquad (1)$$

where $|t| \leq 1$. By partially differentiating this expression with respect to t obtain the recurrence relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)\,. \qquad (2)$$ **[7 marks]**

Given that $P_0(x) = 1$ and $P_1(x) = x$, use this recurrence relation to obtain expressions for $P_2(x)$ and $P_3(x)$. **[4 marks]**

By differentiating expression (1) with respect to x show that:

$$P_n(x) = \frac{dP_{n+1}(x)}{dx} + \frac{dP_{n-1}(x)}{dx} - 2x\frac{dP_n(x)}{dx}\,. \qquad (3)$$ **[5 marks]**

By differentiating (2) and substituting into (3) show that

$$\frac{dP_{n+1}(x)}{dx} = (n+1)P_n(x) + x\frac{dP_n(x)}{dx}\,.$$ **[4 marks]**

4. (a) The matrices $\underline{A}$, $\underline{B}$, and $\underline{D}$ are related by $\underline{D} = \underline{A}\,\underline{B}$. Given that

$$\underline{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 1 \\ 3 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \underline{D} = \begin{pmatrix} 3 & 1 & -1 \\ 7 & -1 & 0 \\ 3 & 9 & 5 \end{pmatrix},$$

evaluate $\underline{A}^{-1}$. **[7 marks]**

Hence derive the value of $\underline{B}$. **[3 marks]**

(b) The function $U(x,t)$ satisfies the differential equation

$$\left(\frac{\partial^2 U}{\partial t^2}\right) + c^2\,U = \left(\frac{\partial^2 U}{\partial x^2}\right),$$

where c is a real constant. By seeking a solution of the equation in the separable form $U(x,t) = X(x) \times T(t)$, show how one obtains a solution for $X(x)$ of the form:

$$X = A\cos kx + B\sin kx\,.$$

[4 marks]

Hence find the most general solution for which $U(0,t) = 0$, $U(L,t) = 0$, and $U(x,0) = 0$. **[6 marks]**

5. Give an example of a second-order, homogeneous, ordinary differential equation which **cannot** be solved using a series expansion. [3 mark]

Write the differential equation

$$2x(x-1)\frac{d^2y}{dx^2} + (6x-1)\frac{dy}{dx} + 2y = 0.$$

in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,.$$

For what values of x are $p(x)$ or $q(x)$ singular? [4 marks]

The equation has a series solution of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,.$$

Write down the indicial equation and hence show that $k = 0$ or $\frac{1}{2}$. How many independent series solutions are there for this problem? [3 marks]

For the series with $k = 0$ show that

$$a_{n+1} = \frac{2(n+1)}{(2n+1)}a_n\,.$$ [5 marks]

Hence obtain the first four terms of the series. [3 marks]

Explain for what values of x you would expect this series solution to be convergent. [2 marks]

6. If f is a scalar function and $\underline{v}$ is a vector function, give the expressions for (i) $\underline{\nabla} f$, (ii) $\underline{\nabla}\cdot\underline{v}$ and (iii) $\underline{\nabla}\times\underline{v}$ in Cartesian coordinates. [7 marks]

Using Cartesian coordinates, or otherwise, show that

$$\underline{\nabla}\cdot(f\,\underline{v}) = \underline{v}\cdot\underline{\nabla} f + f\,\underline{\nabla}\cdot\underline{v}\,.$$ [3 marks]

For the vector $\underline{v} = xyz\,\underline{r}$,where $\underline{r} = x\hat{\underline{e}}_1 + y\hat{\underline{e}}_2 + z\hat{\underline{e}}_3$., evaluate (a) $\underline{\nabla}\cdot\underline{v}$ and (b) $\underline{\nabla}\times\underline{v}$. [5 marks]

Show that this vector satisfies the expression

$$\nabla\times(\nabla\times\underline{v}) = \nabla(\nabla\cdot\underline{v}) - \nabla^2\underline{v}$$ [5 marks]

7. The function $f(x)$ is periodic with period 2π. It takes the value $f(x) = x$ in the range $-\pi \leq x < \pi$. Sketch the function between -2π and 2π. Is the function odd or even? **[3 marks]**

If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, ,$$

show from first principles that the Fourier coefficients are given by

$$\begin{aligned} a_n &= 0 \, , \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \sin nx \, dx \, . \end{aligned}$$

[7 marks]

Assuming the integral

$$\int x \sin nx \, dx = \frac{1}{n^2} \sin nx - \frac{x}{n} \cos nx \, ,$$

evaluate the coefficient b_n, and show that the Fourier series for $f(x)$ is

$$f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx \, .$$

[4 marks]

By considering the series at $x = \frac{\pi}{2}$ show that

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} \, .$$

[4 marks]

Comment on the value of the series in the region $x = \pi$. **[2 marks]**

END OF PAPER