UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Phys & Astro 2246: Mathematical Methods III

COURSE CODE	:	**PHAS2246**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

05-C1028-3-150

TURN OVER

Answer FIVE questions only.

The following series expansions may be useful

$$\ln(1+p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} \ldots (-1)^{n+1}\frac{p^n}{n} \ldots$$

$$\frac{1}{(1+p)^m} = 1 - mp + \frac{m(m+1)}{2!}p^2 - \frac{m(m+1)(m+2)}{3!}p^3 \ldots$$

Numbers in square brackets show the provisional allocation of marks per sub-section of the question.

1. A matrix $\underline{A}$ is said to have an inverse $\underline{A}^{-1}$; what conditions must $\underline{A}$ satisfy for $\underline{A}^{-1}$ to exist? **[2 marks]**

For the matrix

$$\underline{B} = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix},$$

calculate $\underline{B}^2$ and $\underline{B}^3$. Hence show that for non-negative integers n,

$$\underline{B}^{2n} = 2^n \underline{I},$$

$$\underline{B}^{2n+1} = 2^n \underline{B},$$

where $\underline{I}$ is the 2×2 unit matrix. **[5 marks]**

Consider the matrix

$$\underline{A} = \underline{I} + \alpha \underline{B},$$

where α is a real number. Calculate the inverse $\underline{A}^{-1}$ by a direct method. Is this valid for all values of α? **[5 marks]**

By expanding the right hand side of

$$\underline{A}^{-1} = (\underline{I} + \alpha \underline{B})^{-1}$$

as a power series in α, show that

$$\underline{A}^{-1} = \underline{I} - \frac{\alpha}{1-2\alpha^2}\underline{B} + \frac{\alpha^2}{1-2\alpha^2}\underline{B}^2 .$$

For which values of α is your derivation valid?. **[6 marks]**

Show that this result is the same as that obtained by direct inversion. **[2 marks]**

2. What is a Hermitian matrix? Prove that the eigenvalues of a Hermitian matrix are real. **[4 marks]**

A matrix is given by

$$\underline{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & 1 \end{pmatrix} .$$

Is this matrix Hermitian? Demonstrate that the eigenvalues λ of $\underline{A}$ satisfy the characteristic equation

$$\lambda^3 - 9\lambda = 0 .$$

Show that one eigenvalue is $\lambda = 0$ and hence obtain the other eigenvalues for $\underline{A}$. **[7 marks]**

Show that the normalised eigenvector associated with $\lambda = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Find the other two eigenvectors normalised to unit length. **[7 marks]**

For one pair of eigenvectors, demonstrate that they are orthogonal. **[2 marks]**

3. The terms orthogonal and normalised are used for both vectors and polynomial functions. Explain what is meant when a set of vectors and a set of functions are said to be orthogonal and normalised. **[6 marks]**

The generating function for the Legendre polynomials is

$$g(x,t) \equiv (1 - 2x\,t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n ,$$

where $|t| \leq 1$. By considering the integral

$$\int_{-1}^{+1} [g(x,t)]^2\, dx$$

obtain the normalisation condition for a general Legendre polynomial, $P_n(x)$. **[7 marks]**

By expanding $g(x,t)$ as a power series in t, obtain the explicit form for the first three Legendre polynomials, P_0, P_1 and P_2. **[7 marks]**

4. The temperature $T(x, y)$ in a semi-infinite slab, $0 \leq x < \infty$, $-\infty < y < \infty$ satisfies the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \, ,$$

with the boundary conditions

$$\left(\frac{\partial T(x, y)}{\partial y} \right) = 0$$

at $y = 0$, and $T \to 0$ as $x \to \infty$.

By looking for a solution in separable form, $T(x, y) = X(x) \times Y(y)$, show that one possible solution which satisfies the boundary conditions is

$$T(x, y) = e^{-kx} \cos ky \, .$$

Hence show that a more general solution is **[9 marks]**

$$T(x, y) = \int_0^\infty a(k) e^{-kx} \cos ky \; dk \, .$$

where $a(k)$ is a suitably behaved function of k. **[1 mark]**

An even function $f(y)$ and its Fourier cosine transform $g(k)$ are connected by the relations

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(k) \cos ky \; dk, \qquad g(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \cos ky \; dy \, .$$

If $g(k) = p\sqrt{2\pi} e^{-pk}$, with $p > 0$, use the relationship $\cos ky = \frac{1}{2}(e^{iky} + e^{-iky})$ to show that

$$f(y) = \frac{p^2}{p^2 + y^2} \, .$$ **[3 marks]**

Given the extra boundary condition that

$$T(0, y) = \frac{p^2}{p^2 + y^2} \, ,$$

determine $a(k)$ and hence show that the for general values of x

$$T(x, y) = \frac{p(p + x)}{(p + x)^2 + x^2}$$ **[7 marks]**

5. Give an example of a third-order, homogeneous, ordinary differential equation. [3 mark]

Write the differential equation

$$x \frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0$$

in the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,.$$

Using this form explain:

(a) How many series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,,$$

you expect to satisfy this equation;

(b) What you expect the radius of convergence of the series to be. [5 marks]

Show that the only series solution has $k = 0$. Derive the recurrence relation

$$\frac{a_{n+1}}{a_n} = \frac{n-3}{(n+1)^2}\,.$$ [6 marks]

Explain why this recurrence relation leads to a simple polynomial solution and write down this solution. [4 mark]

Demonstrate explicitly that this solution satisfies the original differential equation. [2 marks]

6. The even function $f(x)$ is defined in the range $0 < x < \pi$ as

$$f(x) = \begin{cases} 1, & \text{if} \quad 0 < x < \tfrac{1}{2}\pi \,, \\ 0, & \text{if} \quad \tfrac{1}{2}\pi < x < \pi \,. \end{cases}$$

Sketch $f(x)$ over the range $-\pi < x < \pi$. **[2 marks]**

If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \,,$$

use the relationships

$$\int_{-\pi}^{+\pi} \cos mx \cos nx dx = \int_{-\pi}^{+\pi} \sin mx \sin nx dx = \pi \delta_{m,n}$$

to show from first principles that the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx \, dx \,, \\ b_n &= 0 \,. \end{aligned}$$

[7 marks]

Evaluate the coefficient a_n and show that the Fourier series for $f(x)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos\{(2m+1)x\} \,.$$ **[7 marks]**

Use this expression to find the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$$ **[4 marks]**

7. Using Cartesian coordinates, or otherwise, show that

$$\underline{\nabla} \times (\underline{\nabla}\phi) = 0$$ **[3 marks]**

$$\underline{\nabla} \cdot (\phi\,\underline{S}) = \phi\,(\underline{\nabla} \cdot \underline{S}) + (\underline{\nabla}\phi) \cdot \underline{S}\,,$$ **[3 marks]**

$$\underline{\nabla} \times (\phi\,\underline{S}) = \underline{\nabla}\phi \times \underline{S} + \phi\,(\underline{\nabla} \times \underline{S})\,,$$ **[4 marks]**

where ϕ is a continuous, scalar function and $\underline{S}$ a vector function.

In terms of the radius vector $\underline{r} = x\,\hat{\underline{e}}_x + y\,\hat{\underline{e}}_y + z\,\hat{\underline{e}}_z$, the potential V due to an electric dipole moment $\underline{m}$ (a constant vector) is given by

$$V = \frac{1}{4\pi\varepsilon_0}\,\frac{\underline{m} \cdot \underline{r}}{r^3}\,.$$

Using Cartesian coordinates, or otherwise, show that the resulting electric field $\underline{E} = -\underline{\nabla}V$ is given by

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\underline{m} \cdot \underline{r})\underline{r} - r^2\underline{m}}{r^5}\right)\,.$$ **[5 marks]**

Verify that $\underline{\nabla} \cdot \underline{E} = 0$. **[5 marks]**

END OF PAPER