UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Statistical Thermodynamics

COURSE CODE	:	**PHAS2228**
UNIT VALUE	:	**0.50**
DATE	:	**11-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

06-C1108-3-140

Answer ALL SIX questions from Section A, and THREE questions from Section B

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

You may assume the following (using standard notation):

Mass of the hydrogen atom:	m_H	=	1.67×10^{-27} kg
Mass of the electron:	m_e	=	9.11×10^{-31} kg
Charge of the electron:	e	=	1.60×10^{-19} C
Boltzmann constant:	k	=	1.38×10^{-23} J K^{-1}
Planck constant:	h	=	6.63×10^{-34} J s
Gravitational constant:	G	=	6.67×10^{-11} N m^2 kg^{-2}
Speed of light:	c	=	3.00×10^8 m s^{-1}
Gas constant:	R	=	8.31 J mol^{-1} K^{-1}

Single-particle density of momentum states: $g(p)dp = (V/h^3)4\pi p^2 dp$

SECTION A

1. For a classical gas, the mean number of particles per (one-particle) state, as a function of translational energy, E_{tr}, is given by

$$f_{MB}(E_{tr}) = \frac{N}{V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2} e^{-E_{tr}/kT}$$

where N is the number of atoms, V is the volume of the gas, and T is the temperature.

What is the criterion for a quantal gas to be classical? State, without derivation, what this implies about the mean particle separation. [3]

The mass, m, of a water molecule is 3.0×10^{-26}kg. Show that for $T = 280$K the classical regime is valid for liquid water, for which $N/V = 3.3 \times 10^{28}\text{m}^{-3}$. [4]

2. State the *Third Law of Thermodynamics* (a) in terms of entropy, and (b) in terms of the attainability of absolute zero. Explain (with the help of a diagram) why statement (b) follows from statement (a). [6]

3. Define the meanings of *collision mean free path*, l, and the *collision cross section*, σ_c.

Write down an expression relating, l, σ_c and the mean number of scatterers per unit volume, n_0. [3]

A beam of photons, of intensity I_0, passes through a column of gas, of length 2m and density 7×10^{23} molecules m^{-3}. If the photon absorption cross-section of an individual gas molecule is $5 \times 10^{-24} m^2$, calculate the intensity of the emergent beam. [3]

4. Define the *Helmholtz Free Energy*, F, and explain to what sort of system it applies. [2]
Give an expression which relates F to the internal energy, E, the entropy, S, and the temperature, T. What happens to F for a system in equilibrium ? [2]

Pressure is related to the Helmholtz free energy by:

$$P = -\left(\frac{\partial F}{\partial V}\right)_T .$$

In the classical regime the partition function of a single gas atom of mass m in a volume V is given by $Z(1, V, T) = V(2\pi mkT/h^2)^{3/2}$. Show that for one atom: $PV = kT$ [3]

5. A simple magnetic dipole has spin 1/2 and magnetic moment $\mu = 9.0 \times 10^{-24} JT^{-1}$. The dipole is in thermal equilibrium at T = 4K, in an applied magnetic field B = 2T. The dipole can align itself either parallel or anti-parallel to the magnetic field, with an interaction energy of -μB or +μB respectively. For this dipole calculate:

(a) the partition function

(b) the average interaction energy of the dipole (E_{int}) with the applied field. [5]

Sketch E_{int} as a function of $\mu B/(kT)$. [2]

6. Define what is meant by (a) a *heat bath*, and (b) *adiabatic* changes. Write down an expression for the *First Law of Thermodynamics* for infinitessimal reversible changes, defining the terms. Given that the equation of state for an adiabatic system is given by [4]

$$PV^{\gamma} = constant$$

where γ is a constant, show that the change in the internal energy of such a system in going from state 1 (P_1, V_1) to state 2 (P_2, V_2) is given by [3]

$$\Delta E = \frac{1}{(\gamma - 1)} [P_2 V_2 - P_1 V_1] .$$

SECTION B

7. The Van der Waals equation of state for real gases is given by

$$\left(P + \frac{N^2 a}{V^2}\right)(V - Nb) = NkT.$$

(a) Define all of the terms in this expression and explain what physical effects distinguish this expression from the equation of state for a perfect gas. [3]

With the aid of a diagram, give the mathematical definition and explain the physical significance of the critical point (at which $T = T_c$). [6]

(b) Hence show that, at the critical point, the temperature (T_c), pressure (P_c) and volume (V_c) are related by [7]

$$\frac{T_c}{P_c V_c} = \frac{8}{3Nk}$$

Sketch and label a P, T *phase diagram* appropriate to a simple substance. [4]

8. (a) Explain what is meant by the term *black-body.* [2]

For a single particle in a volume V, the number of allowed momentum states in the range p to $p + dp$ is given by $g(p)dp$ in the rubric. Use this to derive $g(\nu)d\nu$ for a gas of photons.

Given the fact that photons obey Bose-Einstein statistics, derive *Planck's radiation law* for the distribution of *radiation energy density* within a cavity in the frequency range ν to $\nu + d\nu$: [8]

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu/kT) - 1}.$$

(b) Given that the *spectral emittance* of a black-body

$$I_{BB}(\nu, T) = \frac{c}{4} u(\nu, T),$$

show that the *total emittance* is given by

$$I_{BB}(T) = \sigma T^4,$$

and calculate the value of the *Stefan-Boltzmann constant*, σ. [6]
(you may need to use $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$)

(c) A planet is orbiting a star at a distance of 8×10^{11}m. The total radiant power of the star is 3×10^{28}W. If the average absorptivity of the planet is $\bar{\alpha} = 0.7$ and its emissivity $\epsilon = 0.9$ calculate its equilibrium temperature (which you may assume to be uniform). [4]

9. (a) Define the terms *macrostate* and *microstate* in statistical thermodynamics. State the *general* definition of entropy, defining the terms. [4]

(b) An isolated system in a heat bath undergoes a small change in energy, so that

$$dE = \sum_r E_r dp_r + \sum_r p_r dE_r$$

where p_r is the probability that the system is in microstate r. Use this together with the definition of the partition function, Z, to show that the change in energy for infinitessimal changes is given by: [8]

$$dE = TdS - PdV.$$

Hence show that for *infinitessimal, reversible changes*: [2]

$$dS = \frac{đQ}{T}$$

(c) One mole of a perfect gas is compressed isothermally and reversibly from a volume of 1m^{-3} to 0.2m^{-3}. By how much does its entropy change? If the temperature of the heat bath is 300K, calculate (a) the work done on the gas, and (b) the total change in entropy of the (system + heat bath). [6]

10. (a) Consider an isolated system divided into subsystems 1 and 2 by a diathermal partition which is fixed and non-porous. Obtain the condition for thermal equilibrium and show that the temperature T_1 of subsystem 1, which has internal energy E_1, volume V_1, particle number N_1, and entropy S_1, may be defined as [7]

$$\frac{1}{T_1} = \left[\frac{\partial S_1}{\partial E_1}\right]_{V_1,N_1}$$

(b) By considering a similar isolated system, but now with subsystems 1 and 2 separated by *(i)* a moveable, diathermal, non-porous partition, and *(ii)* a fixed partition which is diathermal and porous, derive corresponding expressions for the pressure P and chemical potential μ as derivatives of entropy. [7]

(c) By applying the Clausius Principle show that the definitions of temperature and pressure are consistent with the requirements for heat flow between systems out of thermal balance and volume changes for systems out of pressure balance. [6]

11. (a) Write down an expression for the mean number of particles per (one-particle) state at energy E, $f(E)$, for systems that obey *Fermi-Dirac* statistics and sketch the form of $f(E)$ as a function of E for the cases of low and high temperatures. **[4]**

Explain the meaning of the *Fermi Energy* (E_F) and the *Fermi Temperature* and state under what conditions a fermion gas can be described as *extremely degenerate*? **[3]**

(b) If the electron pressure in a degenerate ideal Fermi gas is given by $P_e = N\langle pv\rangle/3V$ where $\langle pv\rangle$ is the average of the product pv and p,v are the electron momentum and velocity respectively, show that the electron pressure and the Fermi momentum are related by

$$P_e = \left(\frac{8\pi}{15mh^3}\right) p_F^5 \text{ and } P_e = \left(\frac{2\pi c}{3h^3}\right) p_F^4,$$

in the *non-relativistic* and *relativistic* regimes respectively. **[6]**
(You may need to use a modified form of the density of momentum states, $g(p)$, given in the rubric.)

(c) Use this result together with the fact that $E_F \propto (N/V)^{2/3}$ to show that in the *relativistic* regime, $P_e \propto \rho^{4/3}$, where ρ is the mass density. Hence show that in this regime the mass of a white dwarf star is not a function of its radius. Explain the physical significance of this result. **[7]**