Queen Mary
University of London

M.Sc. EXAMINATION

MTHM 033 Relativity and Gravitation

Tuesday, 16 May 2006 14:30-17:30 Pm
Duration: 3 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following:

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ N m^2 kg^{-2}
Speed of light	c	$= 3 \times 10^8$ m s^{-1}
1 kpc		$= 3 \times 10^{19}$ m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+---)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$u^i = dx^i/ds$ is the 4-velocity along the geodesic.

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \quad \text{where} \quad R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R_i^i.$$

Einstein equations:

$$R_k^i - \frac{1}{2}\delta_k^i R = \frac{8\pi G}{c^4} T_k^i,$$

where T_k^i is the Stress-Energy tensor.

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot) \text{ km}, \quad \text{where } M_\odot \text{ is the mass of Sun.}$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2}) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+ \frac{2 r_g r a}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Hamilton-Jacobi equation:

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} - m^2 c^2 = 0,$$

where four-momentum $p_i = -\frac{\partial S}{\partial x^i}$ and $p_0 = E$ is the energy and $p_3 = L$ is the angular momentum.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R} \frac{d^2 D_{\alpha\beta}}{dt^2}, \quad \text{where } D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) dM \text{ is the quadrupole tensor.}$$

SECTION A

Each question carries 8 marks. You should attempt all questions.

1. (a) [**5 Marks**] Show that the redshift $z = \frac{\lambda'}{\lambda} - 1$ of a photon emitted with frequency $\omega = \frac{2\pi}{\lambda}$ at the bottom of the tower of height h and detected with frequency $\omega' = \frac{2\pi}{\lambda'}$ at its top is $z = gh/c^2$.

(b) [**3 Marks**] A rocket of height H takes off with acceleration $10g$ from the surface of the Earth. Apply the equivalence principle to show that the redshift of a photon emitted at the bottom of the rocket and detected at its top is $z = 11Hg/c^2$.

2. (a) [**3 Marks**] Give the definition of the mixed tensor of the $(n+k)$-th rank $A^{i_1\, i_2\, \cdots\, i_n}_{j_1\, j_2\, \cdots\, j_k}$ in terms of the transformation matrices $\alpha^l_m = \frac{\partial x^l}{\partial x'^m}$ and $\beta^l_m = \frac{\partial x'^l}{\partial x^m}$.

(b) [**5 Marks**] In a locally galilean frame of reference $(ct^{(0)}, x^{(0)1},\ x^{(0)2},\ x^{(0)3})$ a mixed tensor of the second rank is defined as $A^{(0)i}_k = \delta^i_0 \delta^1_k$. Using coordinate transformation from the locally galilean frame to the uniformly accelerated non-inertial frame of reference $(ct, x^1,\ x^2,\ x^3\ )$ given by $ct = ct^{(0)}$, $x^1 = x^{(0)1} + \frac{1}{2}at^2$, $x^2 = x^{(0)2}$, $x^3 = x^{(0)3}$, show that

$$A^i_k = A^{(0)i}_k + \frac{at}{c}\delta^i_1 \delta^0_k.$$

3. (a) [**3 Marks**] Show that the Christoffel symbols Γ^i_{kl} are symmetric with respect to their low indices.

(b) [**5 Marks**] Prove that $Dg_{ik} = g_{ik;l}dx^l = 0$ and hence show that Christoffel symbols are

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

4. (a) [**2 Marks**] Show that $u_i u^i = 1$, where u^i is the four-velocity of a particle.

(b) [**6 Marks**] Show that in a static gravitational field with metric

$$ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta,$$

the components of the four velocity are

$$u^0 = \frac{1}{\sqrt{g_{00}\left(1 - \frac{v^2}{c^2}\right)}} \quad \text{and} \quad u^\alpha = \frac{v^\alpha}{c\sqrt{1 - \frac{v^2}{c^2}}},$$

where

$$v^\alpha = \frac{c\,dx^\alpha}{\sqrt{g_{00}}dx^0} \quad \text{is the three velocity and} \quad v^2 = -g_{\alpha\beta}v^\alpha v^\beta.$$

[Next question overleaf.]

5. (a) **[3 Marks]** Using the Einstein Equations show that

$$R_i^k = \frac{8\pi G}{c^4}\left(T_i^k - \frac{1}{2}T\right), \quad \text{where } \ T = T_m^m.$$

(b) **[5 Marks]** Show that for an arbitrary contravariant vector A^i the covariant vector

$$B_i = A^n{}_{;n;i} - A^n{}_{;i;n}$$

can be expressed in terms of the energy-momentum tensor T_k^i as follows

$$B_i = \frac{4\pi G}{c^2} g_{in}\left(TA^n - 2T_m^n A^m\right).$$

6. (a) **[4 Marks]** A gravitational field is described by the metric

$$ds^2 = e^{2t}\eta_{ik}dx^i dx^k.$$

Show that all non vanishing components of the Cristoffel symbol can be presented in the following form

$$\Gamma^i_{nk} = \delta^0_n\delta^i_k + \delta^0_k\delta^i_n - \delta^i_0\eta_{kn}.$$

(b) **[4 Marks]** Show that the scalar curvature R of the above field can be presented as

$$R = e^{-2t}C, \ \text{ where } \ C \ \text{ is a constant scalar.}$$

7. (a) **[4 Marks]** Using the Kerr metric explain why the surface $g_{00} = 0$ is called the limit of stationarity and find the location of the limit of stationarity, r_{st}.

(b) **[4 Marks]** Using the Kerr metric explain why the surface $g^{11} = 0$ is called the event horizon and find the location of the event horizon, r_{hor}.

SECTION B

Each question carries 22 marks. Only marks for the best TWO questions will be counted.

1. (a) **[8 Marks]** Using the coordinate transformations

$$c\tau = ct + \int \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}\left(r - r_g\right)}$$

show that the Schwarzschild metric takes the form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

[This question continues overleaf . . .]

(b) **[6 Marks]** Expressing r in terms of $R - c\tau$, demonstrate that the latter metric is non-stationary. What can be said about the true character of the Schwarzschild space-time metric at $r = r_g$.

(c) **[8 Marks]** Consider the propagation of radial light signals in the above metric. Show that the equation $ds = 0$ for $\theta,\ \phi =$ constant gives

$$\frac{d(c\tau)}{dR} = \pm\sqrt{\frac{r_g}{r}}.$$

Using the $(c\tau, R)$ diagram and the above formula, determine the location of the event horizon and hence demonstrate that the event horizon is a null surface.

2. Consider the propagation of a photon in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzshild gravitational field.

(a) **[2 Marks]** Demonstrate how the Eikonal equation

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k} = 0$$

can be obtained from the Hamiltom-Jacobi equation.

(b) **[10 Marks]** Given that the solution of the Eikonal equation can be written in the following form

$$\Psi = -\omega t + \frac{\omega\rho}{c}\phi + \Psi_r(r),$$

where ω is the frequency of the photon and ϱ is its impact parameter, find a differential equation for Ψ_r and show that

$$\left(1 - \frac{r_g}{r}\right)^{-1}\frac{dr}{cdt} = \sqrt{1 - \frac{\varrho^2}{r^2} + \frac{\varrho^2 r_g}{r^3}}.$$

(c) **[10 Marks]** Sketch the regions of possible motions on the $(r - \rho)$ diagram and hence show that the radius of the unstable stable circular orbit for photons corresponds to $\varrho = \frac{3\sqrt{3}}{2}r_g$ and $r = \frac{3}{2}r_g$.

3. (a) **[8 Marks]** Derive the geodesic deviation equation

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm}u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

 [This question continues overleaf . . .]

(b) **[10 Marks]** Consider two neighboring particles freely falling from rest in the Schwarzschild gravitational field in the same radial direction. Using the geodesic deviation equation show that the component of the Riemann tensor which is responsible for the tidal force in the radial direction is

$$R^1_{001} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

(c) **[4 Marks]** If the height of an observer is $l \approx 2$m, find the radial distance $r \gg r_g$ from a solar mass neutron star at which the radial tidal 3-acceleration experienced by the observer at rest ($a = c^2 \frac{D^2\eta^1}{ds^2}$) is equal to 100 $g \approx 10^3$ms^{-2}. You may assume that the observer's body is aligned along the radial direction and you may take the gravitational radius of the Sun to be 3 km.

4. (a) **[4 Marks]** Consider a plane gravitational wave propagating along the x-axis. All components of $h_{ik} = g_{ik} - \eta_{ik}$ vanish except $h_{22} = -h_{33} \equiv h_+$ and $h_{23} = h_{32} = h_\times$. Let two test particles be located in the $(y-z)$ plane and separated by the 3-vector $l^\alpha = (0, l_0 \cos\theta, l_0 \sin\theta)$. Show that the perturbation of the distance δl between the two particles in the gravitational wave varies as

$$\delta l = l - l_0 = \frac{l_0}{2}(h_+ \cos 2\theta + h_\times \sin 2\theta).$$

(b) **[10 Marks]** Consider a ring of test particles initially at rest in the $(y - z)$ plane and a plane monochromatic gravitational wave with frequency ω and polarization $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$. Sketch the shape of the ring perturbed by the gravitational wave at times $t = \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$. Repeat the analysis for a gravitational wave with another polarization: $h_+ = 0$, $h_\times \sin\omega(t - x/c)$. Finally consider the superposition of two polarized waves: $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = h_0 \cos\omega(t - x/c)$. What would you call this state of polarization?

(c) **[8 Marks]** Consider a binary system located in the center of our Galaxy ($R \approx 10kpc$), and consisting of two components of the same mass m. Show that to an order of magnitude the amplitude of the gravitational radiation generated by the binary and its frequency are $h_0 \sim r_g^2/(rR)$ and $\omega \sim (c r_g^{1/2} r^{-3/2})$ respectively, where r_g is the gravitational radius of each component and r is the separation between the two components. A future gravitational wave antenna detects gravitational radiation with frequency $10^{-3}Hz$ and amplitude 10^{-23}. Estimate the mass m and r.

[End of examination paper.]