Queen Mary
University of London

B.Sc. EXAMINATION

MAS322 Relativity

30 May 2006, Time 10am
Time Allowed: 3 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

You are reminded of the following information, which you may use without proof.

Latin indices run from 0 to 3. Greek indices run from 1 to 3.

The metric tensor of special relativity is η_{ab} such that

$$ds^2 = \eta_{ab}dx^a dx^b = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

The Lorentz transformations between two frames F and F' in standard configuration are

$$x' = \gamma(x - Vt), \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right), \quad y' = y, \quad z' = z$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$ and F' is moving with speed V relative to F.

Partial derivatives:

$$Q_{,a} = \frac{\partial Q}{\partial x^a}$$

Covariant derivatives are denoted by semicolon: e.g. $V_{a;b} = V_{a,b} - \Gamma^c{}_{ab}V_c$

The Riemann curvature tensor:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}$$

Euler–Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

Geodesic equation:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

SECTION A: *You should attempt all questions. Marks awarded are shown next to the questions.*

1. Define the 4-velocity $\bar{v}$ and the 4-momentum $\bar{p}$ for a particle in special relativity.

Prove that $\bar{p}.\bar{p} = -m_0^2$, where m_0 is the rest mass of the particle.

Show that the 4-momentum can be expressed in terms of the 3-velocity of the particle, $\underline{v}$, such that $\bar{p} = m_0\gamma(v)(1, \underline{v})$, where $\gamma(v) = (1 - v^2)^{-1/2}$ and $v = |\underline{v}|$. [In this question you may set $c = 1$.] [12]

2. A train travels into a tunnel at a speed v. An observer at rest inside the tunnel measures the tunnel to have a length L. Derive an expression that relates the length of the tunnel as seen by such an observer to the length as measured by the driver of the train. Determine the speed at which the train must be travelling in order for the driver's measurement of the length of the tunnel to be shorter than L by a factor of five. [8]

3. Show that an antisymmetric contravariant tensor S^{ab} remains antisymmetric under a general coordinate transformation.

Determine the value of $P_{abc}Q^aQ^bQ^c$, where P_{abc} is an arbitrary tensor that is antisymmetric over its indices a and b and Q^a is an arbitrary contravariant vector. [10]

4. Explain what is meant by a Local Inertial Frame. Suppose that in such a frame, an arbitrary contravariant tensor, V^a, satisfies the condition

$$V^a{}_{,b} = 0$$

What can you say about the value of the covariant derivative of this tensor? Explain your reasoning. [10]

5. Consider the three-dimensional Riemannian manifold with coordinates (x, y, z) and a metric (line-element) of the form

$$ds^2 = dx^2 + x^3dy^2 + y^2dz^2$$

[This question continues overleaf . . .]

Calculate the covariant (g_{ab}) and contravariant (g^{ab}) components of the metric tensor for this spacetime.

The components of the connection are given by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left[g_{db,c} + g_{dc,b} - g_{bc,d} \right]$$

Employ this formula to calculate the connection components $\Gamma^x{}_{xx}$, $\Gamma^x{}_{yy}$ and $\Gamma^y{}_{zz}$. [12]

6. Write down the Einstein field equations in the standard form in terms of the Ricci tensor, the Ricci scalar and the energy-momentum tensor.

Suppose that the components of the Ricci tensor are proportional to the corresponding components of the metric tensor, i.e., that $R_{ab} = \Lambda g_{ab}$, where Λ is an arbitrary constant of proportionality. Show that this implies that the energy-momentum tensor must also be proportional to the metric tensor and determine the constant of proportionality in terms of Λ. [8]

SECTION B: *Each question carries 20 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.*

7. A particle with a rest mass m_1 moves with a speed u_1 along the x-axis. It collides with a particle with rest mass m_2 that is travelling in the opposite direction with a speed u_2. The particles coalesce to form a single particle with rest mass M that moves along the x-axis with speed u.

Derive an expression for the rest mass of the new particle in terms of m_1, m_2, u_1 and u_2 and verify that $M > m_1 + m_2$.

Suppose the two colliding particles have different rest masses but travel at the same speed, $u_1 = u_2$. Show that the particle that is formed in the collision will travel in the same direction as that of the colliding particle that has the higher rest mass.

Now suppose that the colliding particles travel at different speeds but have the same rest mass, $m_1 = m_2$. Show that the energy of the particle with speed u_1 before the collision is given by

$$E_1 = \frac{M}{2\gamma(u)[1 - uu_1]}$$

[This question continues overleaf ...]

where $\gamma(u) = [1 - u^2]^{-1/2}$.

[In this question you may set the speed of light $c = 1$.]

8. The metric for a particular two–dimensional Riemannian spacetime is given by

$$ds^2 = -e^{2Ar}dt^2 + dr^2$$

where A is an arbitrary constant.

Employ the geodesic equation to calculate all the components of the connection $\Gamma^a{}_{bc}$ for this metric. Hence calculate the R_{tt} component of the Ricci tensor.

9. Give the definitions of the Ricci tensor, R_{bd}, and Ricci scalar, R, in terms of contractions of the Riemann tensor $R^a{}_{bcd}$.

From the definition of the Riemann tensor given on the front page, prove that

$$R^a{}_{bcd} = -R^a{}_{bdc}$$

Prove that the Ricci tensor (and hence also the Einstein tensor) is symmetric, $R_{bd} = R_{db}$. [You may quote the relation $\Gamma^a{}_{ab,c} = \Gamma^a{}_{ac,b}$ without proof.]

Hence, prove that $R^a{}_{bcd}R^{cd} = 0$.

Finally, starting from the Bianchi identities:

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0$$

show that

$$R^a{}_{e;a} - \frac{1}{2}R_{;e} = 0$$

10. The Schwarzschild spacetime metric given in standard form is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

where G and M are constants.

Show, by applying the Euler-Lagrange equation, that the geodesic equation for the coordinate t can be integrated such that

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = l,$$

[This question continues overleaf ...]

where τ is proper time and l is a constant of integration.

Show further that the equation for a radial, timelike geodesic can be written in the form

$$\left(\frac{dr}{d\tau}\right)^2 = l^2 - 1 + \frac{2GM}{r}$$

Show that the choice $l = 1$ can correspond to placing a particle at infinity with zero initial velocity.

What is the significance of the value $r = 2GM$? Assuming $l = 1$, show that it takes a particle moving on a radial timelike geodesic a finite proper time to fall from some initial value of $r = r_0 > 2GM$ to $r = 2GM$, but an infinite coordinate time.

[In this question you may quote the integral

$$\int dy \, \frac{y^{3/2}}{y-1} = \frac{2}{3}y^{3/2} + 2y^{1/2} - \ln\left(\frac{y^{1/2}+1}{y^{1/2}-1}\right)$$

without proof].

[End of examination paper.]