Queen Mary College

UNIVERSITY OF LONDON

MAS 313 Cosmology

20 May 10:00, 2002

Time Allowed: 2 Hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination.

Physical Constants

Gravitational Constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of Light	c	3×10^8 m s^{-1}
Mass of Electron	m_e	9.1×10^{-31} kg
Mass of Proton	m_p	1.7×10^{-27} kg
Boltzmann's Constant	k_B	1.4×10^{-23} J K^{-1}
Black-body Constant	a	7.6×10^{-16} J K$^{-4}m^{-3}$
Stefan-Boltzman Constant	σ	5.7×10^{-8} W K$^{-4}m^{-2}$

1 Mpc = 3×10^{22} m.
1 Mev = 1.6×10^{-13} J.

The following results may be quoted without proof:

The Robertson–Walker metric is

$$ds^2 = -c^2dt^2 + R^2(t)[d\chi^2 + \left(\frac{\sin A\chi}{A}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2)],$$

where $R(t)$ is the scale factor, $A = \sqrt{k}$, and k takes values 0,1,-1.

Exam paper continued

The dynamical equation governing the evolution of the scale factor (including the cosmological constant terms) is:

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{\Lambda R}{3}.$$

The Friedmann equation is:

$$\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda R^2}{3}.$$

The law of energy conservation is

$$d(\rho c^2 R^3) = -3pR^2 dR.$$

The critical density is

$$\rho_{cr} = 3H_0^2/(8\pi G) = 1.8 \times 10^{-26} h^2 kgm^{-3},$$

and

$$h = H_0/(100 \cdot km\ s^{-1}\ Mpc^{-1}).$$

SECTION A: Each question carries 14 marks. You should attempt ALL questions.

A1

Show that in an infinite static Universe with a uniform density of sources the night sky should be as bright as the Sun (the Olber's paradox).

Imagine a static sphere of radius R, filled with stars of radius $r_s \approx 7 \cdot 10^5 km$, and separated from each other by a distance $D = 1\ pc$. Assume that we live in a Universe which is such a sphere with nothing around it, and the brightness of all stars in such a Universe is exactly equal to the brightness of the Sun. Find the radius R of such a Universe, assuming that the brightness of the sky at night is 10^6 times less than the brightness of the Sun.

Explain qualitatively why the evolution and expansion of the Universe can resolve this paradox.

Exam paper continued

A2

Write down the Hubble law in vector form.

Consider three galaxies in an expanding Universe located at points a, b and c. Prove that if the Hubble law is valid for an observer at a, then it is also valid for observers at b and at c.

Assume that the vector $\vec{r}_{ab}$ is perpendicular to the vector $\vec{r}_{ac}$. For an observer in galaxy a galaxy b is redshifted with $z^b_{(a)} = 0.6$ and galaxy c is redshifted with $z^c_{(a)} = 0.8$. Find the redshift $z^c_{(b)}$ of galaxy c, measured by an observer in galaxy b.

A3

Describe briefly the method of determination of distances using supernovae (the Baade-Wesselink method). Estimate to order of magnitude the distance, D, to the galaxy in which a supernova exploded, if at the end of the first day after the explosion of the supernova the expansion speed of the shell $v = 10^3\ km\ s^{-1}$, its temperature $T = 10^4\ K$, and apparent luminosity $l = 5.7 \times 10^{-16}\ W\ m^{-2}$. Give the answer in Mpc.

A4

Assume that the Universe is closed ($k = 1$) and contains only dust ($\alpha = 1$ and $\Lambda = 0$). Using the Friedman equation and the energy conservation equation, demonstrate that the solution of the Friedman equation can be presented in the following parametric form

$$R(\eta) = \frac{\beta}{2}(1 - \cos\eta), \quad t(\eta) = \frac{\beta}{2c}(\eta - \sin\eta),$$

where η is a variable which runs from 0 to 2π and β is some constant.

Express β in terms of the Hubble constant H_0 and the dimensionless density Ω_0, both taken at the present moment of time t_0.

A5

The present Universe contains blackbody radiation with temperature $T = 2.7K$. Estimate the number of photons per baryon, if the matter density parameter is $\Omega_0 = 1$ and the Hubble constant $H_0 = 50\ km\ s^{-1}\ Mpc^{-1}$. Show that this number does not depend on time.

Estimate the density of the matter at the moment when the density of the black body radiation was as high as the density of iron ($7.8\ g\ cm^{-2}$).

Explain qualitatively why the temperature of the neutrino background is lower than the temperature of the microwave background.

Exam paper continued

SECTION B: Each question carries 30 marks. You may attempt all questions but only marks for the best ONE question will be counted.

B1

Using the energy conservation equation, show that if $R = R_0(t/t_0)^\beta$ and $p = \alpha\rho c^2$ the dynamical equation for the scale factor $R(t)$ can be written in the following form

$$\frac{\beta(1-\beta)}{t^2} = \frac{4\pi G\rho_0}{3}(1+3\alpha)\left(\frac{R}{R_0}\right)^{-3(1+\alpha)} - \frac{\Lambda}{3}.$$

Assuming $\alpha \neq -1$ find Λ and β from this equation in terms of α.

Then show that the density ρ always varies in inverse proportional to time unless $\beta \neq 0$ and is equal to

$$\rho(t) = \frac{1}{6\pi G(1+\alpha)^2 t^2}.$$

Consider separately the case $\beta = 0$ (the static Universe) and for this case find Λ in terms of ρ and α.

Starting from the energy conservation equation consider separately the case $\alpha = -1$ and find R, ρ and Λ.

For what value α_* does R grow at the same rate as the particle horizon, ct? Does it mean that the universe is empty?

Consider the case $\alpha = \alpha_*$ separately and find ρ and Λ in this case.

B2

Using the equations governing the evolution of the scale factor, show that the deceleration parameter $q = -R\ddot{R}/\dot{R}^2$ is given by

$$q = \frac{1}{2}\Omega(1+3\alpha) - \frac{\Lambda}{3H^2},$$

where $H = \dot{R}/R$, $\Omega = \rho/\rho_{cr}$, and $\rho_{cr} = 3H^2/(8\pi G)$.

Find q, if the Universe contains i) only dust, ii) only radiation.

Identify a condition that must be satisfied by the equation of state, if $\Lambda = 0$ and q is negative.

Discuss briefly what conclusions we could make if observations showed $q < 0$.

Show that the Friedman equation can be written in the following form

$$H^2 = \frac{8\pi G}{3}\rho + \Lambda/3 - kc^2/R^2.$$

Compare three terms in the Friedman equation: the matter term corresponding to arbitrary α, the Λ-term and the curvature term. Write down the way in which each of these terms behaves as a function of the scale factor R. By considering suitable ranges of values of α, determine conditions for the domination of each term at early times and at late times.

Question continued

The current observations of the black body radiation reveal that the Universe expands with acceleration and this acceleration is substantially determined by the Λ-term. Evaluate the effective dimensionless density Ω_Λ corresponding to this Λ-term (the so called "dark energy"), if as follows from the observations, $\Omega_0 = \Omega_m + \Omega_\Lambda = 1$ and $q = -0.05$. Here Ω_m corresponds to the matter with $\alpha = 0$.

B3

Consider a sphere with $\chi = \chi_s$. Using the Robertson-Walker metric, find

i) the physical radius of the sphere, $r(\chi_s)$,

ii) the circumference of a circle in the equatorial plane ($\theta = \pi/2$), $C(\chi_s)$.

Assume that a spherical galaxy of radius D has redshift z and apparent angular diameter $\Delta\theta$. Assume also that $\alpha = 0$ (dust) and $\Lambda = 0$. Using the equation for radially propagating photons $ds = 0$, determine the relationship between z and χ. Then find $\Delta\theta$ as a function of z in the following two cases: i) a spatially flat Universe, ii) an empty Universe.

Explain qualitatively the physical reasons why the dependence of $\Delta\theta$ on z is non-monotonic and find the value of z_m at which the angular diameter attains its minimum in the cases i) and ii).

Imagine that you observe some galaxy on the sky with known physical diameter D and known redshift z. You can measure its angular diameter $\Delta\theta$ with very high accuracy. What fractional accuracy should you achieve in measurement of the apparent angular diameter of the galaxy in order to determine from your observations whether the Universe is empty or spatially flat? Give a numerical answer to a whole number of percent, if $z = 0.04$.

B4

Derive the equation for the evolution of small density perturbations, $\delta = (\rho' - \rho)/\rho$ after decoupling, to show that

$$\ddot{\delta} + (4/3t)\dot{\delta} - (2/3t^2)\delta = 0.$$

(*Hint: Take into account that* $\rho' R'^3 = \rho R^3$.) Solve this equation using the trial solution $\delta \propto t^m$ to obtain the two modes of perturbations: $\delta = A(t/t_0)^{m_1} + B(t/t_0)^{m_2}$.

According to the COBE observations of the Microwave Background anisotropy, the amplitude of the density perturbations at the moment of decoupling is about 10^{-5} (Take the redshift at this moment to be $z = 999$). Assuming that the first objects were formed at a redshift $z = 9$, estimate the two arbitrary constants in your solution for the density perturbations.

END OF EXAMINATION

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