B.Sc. Examination

MAS 206 Dynamical Astronomy

Thursday 25 May 2000, 10 am

Duration: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

SECTION A: Each question carries 12 marks. You should attempt ALL questions.

A1. A satellite moves in an elliptical orbit about a spherical planet of radius R. When the satellite is 90° from its pericentre, the shortest distance from the centre of the satellite to the surface of the planet is D_1. When the satellite is at its apocentre, the shortest distance to the surface of the planet is D_2. Use the polar equation of a keplerian ellipse to derive expressions for the eccentricity and semimajor axis of the satellite's orbit in terms of D_1, D_2 and R.

A2. Tidal stripping of the stars in a galaxy can occur if it passes sufficiently close to another galaxy. Consider the passage of a small galaxy (mass m and radius R) close to a large galaxy (mass M). Let the radial separation of their centres be r where $r \gg R$, and consider the large galaxy to act as a point mass.

(a) Calculate the magnitude of the gravitational force per unit mass due to the large galaxy at the points on the small galaxy (i) closest to and (ii) furthest from the large galaxy. Hence show that the magnitude of the difference between either of these two quantities and the value at the centre of the small galaxy is given by

$$\frac{\Delta F}{m} \approx 2\frac{\mathcal{G}MR}{r^3}$$

where $\mathcal{G}$ is the universal gravitational constant and terms of second order or higher in R/r have been neglected.

(b) By comparing this difference with the gravitational force per unit mass directed towards the centre of the small galaxy at these points, find the critical value of the separation below which the stars in the small galaxy will be tidally stripped.

A3. Two masses move in circular orbits about their common centre of mass. In the frame rotating at the angular velocity, n, of both masses, the planar equations of motion of a test particle are given by

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y}$$

where x and y are the components of the position vector, and $U = U(x, y)$ is a function of x and y.

(a) Show that the quantity

$$C = 2U - \dot{x}^2 - \dot{y}^2$$

is a constant of the motion.

(b) Use the fact that U is a function of r_1 and r_2 to find expressions for $\partial U/\partial x$ and $\partial U/\partial y$ in terms of $\partial U/\partial r_1$ and $\partial U/\partial r_2$ and hence show that a trivial solution of $\partial U/\partial x = 0$ and $\partial U/\partial y = 0$ is $\partial U/\partial r_1 = \partial U/\partial r_2 = 0$.

A4. The orbit of a planet about a star has semimajor axis a and an orbital period T. Interior to the orbit of the planet are the locations of an infinite sequence of $p+1:p$ resonances where the orbital period, T_p, of a test particle is related to T by the equation $T/T_p = (p+1)/p$, where $p = 1, 2, 3, \ldots$ is an integer. Derive an expression for the separation in semimajor axis between the $p+1:p$ and $p+2:p+1$ resonances as a function of a and p. Show that in the case where p is large this separation is approximately $(2a/3)(1/p^2)$.

A5. A system of N gravitationally interacting masses has a total potential energy $-U$ and a total kinetic energy T. The total energy of the system is E.

(a) If the moment of inertia of the system about a fixed point is constant, state the virial theorem and use it to express (i) E as a function of U and (ii) T as a function of U.

Consider a spherical cluster composed of N stars, each of mass m. Each star has a typical random velocity V and any pair of stars have an average separation R. There are $N(N-1)/2$ possible pairings of stars in the cluster.

(b) Write down the total kinetic energy and the total potential energy of the star cluster and, assuming that the moment of inertia is constant, use the virial theorem to find an expression for V.

SECTION B: Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.

B1. A satellite of mass m moves in a circular orbit of radius a about a planet. A test particle is located at a radial distance r from the centre of the planet and the line from the particle to the planet makes an angle θ with the planet–satellite line. The potential energy per unit mass experienced by the particle due to the satellite is given by $V = -\mathcal{G}m/\Delta$ where Δ is the distance from the particle to the centre of the satellite.

(a) Show that

$$\Delta = a\left[1 - 2\left(\frac{r}{a}\right)\cos\theta + \left(\frac{r}{a}\right)^2\right]^{1/2}.$$

(b) If a body is in hydrostatic equilibrium, what can be said about the gravitational potential on its surface?

(c) Show that in the case where $r \ll a$,

$$V \approx -\frac{\mathcal{G}m}{a}\left[1 + \left(\frac{r}{a}\right)\cos\theta + \left(\frac{r}{a}\right)^2\frac{1}{2}(3\cos^2\theta - 1)\right]$$

where terms of third order and higher in r/a have been neglected. Explain why the term in $(r/a)^2$ gives rise to two tides per day on the planet due to the satellite.

B2. In the two-body problem Kepler's equation, $M = E - e\sin E$, relates the mean anomaly, M, to the eccentric anomaly, E, and the eccentricity, e.

(a) Use a diagram of a keplerian ellipse and a circumscribed circle to show clearly the geometric relationship between E and the true anomaly, f.

(b) A series solution to Kepler's equation can be obtained from the iteration scheme, $E_{i+1} = M + e\sin E_i$, $(i = 0, 1, \ldots)$, where $E_0 = M$ is an initial approximation. Use this scheme with two iterations to show that

$$E = M + e\sin M + \frac{1}{2}e^2\sin 2M + \mathcal{O}(e^3).$$

(c) The true anomaly is related to M and E by the equation

$$f = \sqrt{1-e^2}\int\left(\frac{\mathrm{d}E}{\mathrm{d}M}\right)^2\mathrm{d}M.$$

Use the series solution for E given in part (b) to show that

$$f - M \approx 2e\sin M + \frac{5}{4}e^2\sin 2M + \mathcal{O}(e^3).$$

B3. The torque experienced by a satellite of mass m, moving in a circular orbit of radius a, due to the tidal bulge it raises on a homogeneous planet of radius R is

$$\Gamma = \mathcal{G}\frac{m^2}{a}\left(\frac{R}{a}\right)^5 \frac{3}{2}k_2 \sin 2\theta.$$

where k_2 (a constant) is the Love number of the planet, $\mathcal{G}$ is the universal gravitational constant, and θ is the lag angle.

(a) If E is the sum of the rotational energy of the planet and the orbital energy of the satellite–planet system, show that $\dot{E}$, the rate of change of this energy, is given by

$$\dot{E} = I\Omega\dot{\Omega} + \frac{1}{2}mn^2 a\dot{a}$$

where I is the moment of inertia of the planet, Ω is the rotational frequency of the planet and n is the mean motion of the satellite. Hence show that if the total angular momentum (i.e. rotational plus orbital) of the system is conserved then

$$\dot{E} = -\frac{1}{2}man\dot{a}(\Omega - n).$$

(b) Given that $\dot{E} = -\Gamma(\Omega - n) < 0$, use the results from part (a) to show that the semimajor axis of the satellite will change at a rate given by $\dot{a} \propto a^{-11/2}$, and give the explicit form of the constant of proportionality.

B4. A planet moves in a circular orbit about a central star. In a rotating frame moving with the uniform angular velocity of the planet, the path of the particle moving on an elliptical coplanar orbit interior to that of the planet will have 'loops' for sufficiently large values of the particle's eccentricity, e.

(a) Give a brief, qualitative explanation for these loops, stating whereabouts they occur in the orbit of the particle.

(b) Draw a sketch showing how the appearance of the path of the particle in the rotating frame changes as a function of e for the particular case of the 3:2 resonance.

(c) In the general case of the $p+q:p$ resonance, where p and q are positive integers, show that the loops just start to form when e satisfies the equation

$$(1+e)^3 = [(p+q)/p]^2(1-e).$$

(You may use the fact that the angular momentum per unit mas of the particle is $na^2\sqrt{1-e^2}$ where n, a, and e are the particle's mean motion, semimajor axis and eccentricity.)

End of examination paper.