# Queen Mary, University of London

# MAS 206 Dynamical Astronomy

23 May 2002, 14:30 pm

**Time Allowed: 2 hours**

*This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.*

*Calculators are NOT permitted in this examination.*

*SECTION A: Each question carries 12 marks. You should attempt ALL questions.*

A1. A planet orbits a star on an elliptical path with eccentricity, $e$. At any given time its angular position can be defined either in terms of its true anomaly, $f$, or its eccentric anomaly, $E$.

(a) Sketch a diagram of a Keplerian ellipse and the circumscribed circle and use it to show clearly the angles $f$ and $E$.

(b) The equation of an ellipse with centre at the origin and of eccentricity, $e$, in rectangular coordinates is $(\bar{x}/a)^2 + (\bar{y}/b)^2 = 1$, where $a$ and $b = a\sqrt{1-e^2}$ are the semi-major and semi-minor axes of the ellipse respectively. By deriving expressions for $\bar{x}$ and $\bar{y}$ from your diagram, show that the $x$ and $y$ components of the position vector of the planet, referred to the standard Keplerian coordinate system with origin at the centre of the star, are given by

$$x = a(\cos E - e), \quad y = a\sqrt{1-e^2}\sin E.$$

(c) Show that the radial distance of the planet from the centre of the star is given by

$$r = a(1 - e\cos E).$$

A2. A satellite of mass $m$ moves in a circular orbit of radius $a$ about a planet. A test particle is located at a radial distance $r$ from the centre of the planet and the line from the particle to the planet makes an angle $\theta$ with the planet–satellite line. The square of the distance between the particle and the centre of the satellite is given by

$$\Delta^2 = a^2 - 2ar\cos\theta + r^2.$$

Show that in the case where $r \ll a$ (i.e. the particle is close to the planet) the potential energy per unit mass experienced by the particle due to the satellite is given by

$$V = -\frac{Gm}{a}\left[1 + \left(\frac{r}{a}\right)\cos\theta + \left(\frac{r}{a}\right)^2 \frac{1}{2}(3\cos^2\theta - 1)\right] + \mathcal{O}(r/a)^3$$

where $G$ is the universal gravitational constant. Explain briefly why the term involving $(r/a)^2$ in the above expression for $V$ gives rise to a tidal bulge on the planet.

A3. In the two-body problem, Kepler's equation

$$M = E - e \sin E,$$

relates the mean anomaly, $M$, to the eccentric anomaly, $E$, and the eccentricity, $e$.

(a) Devise an iterative scheme for the *numerical* solution of Kepler's equation which, for given values of $M$ and $e$, relates $E_{i+1}$ (the $(i+1)$th iterate of $E$) to $E_i$.

(b) An analytical approximate solution of Kepler's equation can be obtained for small $e$ by means of a series solution in $M$. Taking $E_0 = M$ as a first approximation, find an expression for the series solution that includes terms up to the second order in $e$.

A4. The equations of motion in the rotating frame of the test particle in the planar, circular restricted three-body problem are given by

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

where $x$ and $y$ are the components of the position vector, $U$ is given by

$$U = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

where ${r_1}^2 = (x + \mu_2)^2 + y^2$, ${r_2}^2 = (x - \mu_1)^2 + y^2$, $\mu_1 = m_1/(m_1 + m_2)$, $\mu_2 = m_2/(m_1 + m_2)$, and the mean motion of of the main 2–body component is taken to be unity. Show that the quantity $C = 2U - \dot{x}^2 - \dot{y}^2$ is a constant of the motion. Define what is meant by an *equilibrium point* of the system. Derive explicit expressions for $\partial U/\partial x$ and $\partial U/\partial y$, and hence show that there are equilibrium points at $x = \frac{1}{2} - \mu_2$ and $y = \pm\sqrt{3}/2$.

A5. Radio observations of atomic hydrogen in the Galaxy allow measurements of the velocity of the gas along the line of sight from our observation point, $S$, which is at a radial distance $r_{\rm s}$ from the centre, $O$, of the Galaxy. The Galaxy can be considered as a coplanar system with material moving in circular orbits about its centre. Consider the motion of a cloud at a point $C$ at a radial distance $r$ from the centre of the Galaxy. Let $l = O\hat{S}C$ be the angle between the centre–Sun and Sun–cloud directions, and $\theta = S\hat{O}C$ be the angle between the centre–Sun and centre–cloud directions. Viewed from the rotating frame of $S$, the contribution to the line of sight velocity, $V_{\|}$, from the cloud is $V_{\|} = r[\Omega(r) - \Omega(r_{\rm s})]\sin(\theta + l)$ where $\Omega(r)$ is the angular velocity at a radial distance $r$.

(a) Sketch the geometry of the system and show that $V_{\|} = r_{\rm s}[\Omega(r) - \Omega(r_{\rm s})]\sin l$.

(b) Show that the circular velocity of the cloud at a distance $r$ is given by

$$V(r) = V_{\rm max} + r_{\rm s}\Omega(r_{\rm s})\sin l$$

where $V_{\rm max}$ is the maximum value of $V_{\|}$ for the cloud.

continued overleaf ...

*SECTION B: Each question carries 20 marks. You may attempt all questions but only marks for the best TWO questions will be counted.*

B1. A binary system consists of two stars of masses $m_1$ and $m_2$ and position vectors $\underline{r}_1$ and $\underline{r}_2$ (with respect to a fixed origin) moving under their mutual gravitational attraction.

(a) Taking $\underline{r} = \underline{r}_2 - \underline{r}_1$ to be the position vector of the mass $m_2$ with respect to $m_1$, write down an expression for the force experienced by each mass. Hence show that $\underline{r}$ satisfies the vector equation of motion

$$\ddot{\underline{r}} + G(m_1 + m_2)\underline{r}/r^3 = 0$$

where $G$ is the universal gravitational constant. Prove that $\underline{h} = \underline{r} \times \dot{\underline{r}}$ is a constant of the motion and use the expressions for $\underline{r}$ and $\dot{\underline{r}}$ in polar coordinates to show that $h = |\underline{h}| = r^2\dot{\theta}$.

(b) Use the expression for $\ddot{\underline{r}}$ in polar coordinates to show that the equation of motion can be written in scalar form as

$$\ddot{r} - r\dot{\theta}^2 = -G(m_1 + m_2)/r^2, \qquad \text{where } r = |\underline{r}|.$$

(c) By making the substitution $u = 1/r$ show that the scalar equation of motion can be written as

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{G(m_1 + m_2)}{h^2}\,.$$

Solve this equation to find $u(\theta)$ and hence $r(\theta)$, relating any constants of integration to the semi-major axis, eccentricity and longitude of pericentre of the orbit of $m_2$ with respect to $m_1$.

[*You may assume that the velocity and the acceleration in polar coordinates are given by* $\dot{\underline{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ *and* $\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + [\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})]\hat{\theta}$ *respectively.*]

B2. Consider a system of $N$ gravitationally interacting particles of masses $m_i$ with position vectors $\underline{R}_i$, relative to a coordinate system with a fixed origin at $O$.

(a) Write down the total potential energy $-U$ and the total kinetic energy $T$ of the system. Let the total energy of the system be $E$.

(b) Starting with the total moment of inertia $I$ of the system about $O$ and differentiating $I$ twice with respect to time, obtain the virial theorem in the form

$$\frac{d^2I}{dt^2} = 4T - 2U\,.$$

Assuming $d^2I/dt^2 = 0$, express (i) $E$ in terms of $U$ and (ii) $T$ in terms of $U$.

(c) Consider the two body problem consisting of a planet of mass $m$ in a circular orbit of radius $r$ about a star of mass $M$. Assuming $m \ll M$, verify the relation in (b) part (i) for $E$ in terms of $U$.

[*You may assume that the magnitude of the velocity of the planet in a general elliptical orbit is given by* $V^2 = G(M+m)(2/r - 1/a)$, *where $a$ is the semi–major axis of the ellipse.*]

B3. The two satellites $S_1$ and $S_2$ around a planet $P$ undergo almost instantaneous changes in the semi-major axes of their orbits every $T$ years when they encounter one another. The encounters are assumed to produce symmetric changes in each orbit such that if the initial semi-major axis of a satellite is $1 \pm \Delta a$ before the encounter, its value after the encounter will be $1 \mp \Delta a$, where $\Delta a \ll 1$.

(a) Assuming (i) that the masses of the two satellites are negligible compared with the mass of the planet, (ii) that there is conservation of the total orbital angular momentum of the ($S_1$–$S_2$) satellite system and (iii) that the satellites always move in circular orbits, show that

$$\frac{M_{\mathrm{s}_1}}{M_{\mathrm{s}_2}} \approx \frac{\Delta a_{\mathrm{s}_2}}{\Delta a_{\mathrm{s}_2}}$$

where $M_{\mathrm{s}_1}$ and $M_{\mathrm{s}_2}$ are the masses of the satellites $S_1$ and $S_2$ and $\Delta a_{\mathrm{s}_1}$ and $\Delta a_{\mathrm{s}_2}$ are the differences of their semi-major axes from unity.

(b) If the mass of $S_2$ is negligible compared to the mass of $S_1$, draw a sketch showing the approximate location of all the Lagrangian equilibrium points in relation to the positions of $S_1$ and the planet. Given that the mass of $S_1$ is less than $10^{-9}$ that of the planet, label each of the points and state which of them are linearly stable and which are unstable.

B4. The orbit of a satellite about a planet has a semi-major axis $a_s$ and an orbital period $T_s$. Interior to the orbit of the satellite are the locations of an infinite sequence of first-order, $q+1 : q$ resonances where the orbital period, $T$, of a test particle is related to $T_s$ by the equation $T_s/T = (q+1)/q$, where $q = 1, 2, 3, \ldots$ is an integer.

(a) If the orbit of the test particle has semi-major axis $a$, where $a < a_s$, write down $a$ as a function of $q$ and $a_s$ for the $q+1 : q$ interior resonance. Hence, by expanding in terms of $1/q$, show that when $q$ is large the separation of adjacent $q+1 : q$ resonances is approximately $(2/3)(1/q)^2 a_s$.

(b) Assume that each resonance has a constant width given by $W = 6\sqrt{m_s/m_p}\, a_s$, where $m_s$ and $m_p$ are the masses of the satellite and planet respectively. Use the expression for the separation given in part (a) to derive a condition for the overlap of adjacent $q+1 : q$ resonances, when $q$ is large. Hence show that if $m_s/m_p = 10^{-8}$ then overlap would occur for all resonances with $q > 33$.

**End of examination paper.**