Answer 1 (i) [6 marks] Recall Gauss' law $\int_V \nabla \mathbf{X}\, dV = \int_S \mathbf{X} \cdot d\mathbf{S}$, for any vector field $\mathbf{X}$. Let C be the closed contour in space with line element $d\mathbf{l}$ along the contour. We also have Stokes' theorem $\oint_C \mathbf{X} \cdot d\mathbf{l} = \int_{S'} (\nabla \times \mathbf{X}) \cdot d\mathbf{S}'$. These results may be used to write Maxwell's equations as given. These follow by integrating the scalar Maxwell equations over a volume V and the vector equations over a surface S' with element $d\mathbf{S}'$.

(ii) [8 marks] Consider first a very small shallow cylinder which straddles the boundary between the two regions, for which the normals to the circular ends of the cylinder are perpendicular to the boundary. Apply the first and third of the Maxwell relations above to the volume and surface of this cylinder. Ignoring the contribution from the infinitely thin sides of the cylinder one finds that $\oint_S \mathbf{D} \cdot d\mathbf{S} = (\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}\Delta a$, where Δa is the area of the circular end of the cylinder, and $\mathbf{n}$ the unit normal to the boundary. For the electric case, given a surface charge density σ we have $\int_V \rho\, dV = \sigma \Delta a$. Thus we deduce the boundary conditions

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma, \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0,$$

the second equation following by similar arguments, noting the absence of magnetic charges.

Now consider a small rectangle which straddles the boundary between the two media. This rectangle has short sides which are infinitesimally small, and longer sides of length Δl which are parallel to the boundary. The unit normal $\mathbf{t}$ to the rectangle is tangent to the interface between the regions. Then $\oint_C \mathbf{H} \cdot d\mathbf{l} = (\mathbf{t} \times \mathbf{n}) \cdot (\mathbf{H}_2 - \mathbf{H}_1)\Delta l$. Assume that there is a current density $\mathbf{K}$ flowing on the rectangle surface $\mathbf{S}'$. Then $\int_{S'} (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S}' = \mathbf{K} \cdot \mathbf{t} \Delta l$, since the $\mathbf{D}$ term vanishes as the area of the cylinder goes to zero. Thus we deduce from the second and fourth of the Maxwell integral relations that

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}, \quad \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0,$$

the second equation following again by the same arguments, noting the absence of magnetic sources.

(iii) [6 marks]

The relevant boundary condition is

$$\mathbf{E}_p + \mathbf{E}''_p = \mathbf{E}'_p,$$

where the subscript p refers to the component parallel to the interface. This immediately gives the required equation. If this is true for all x then the exponents in the equation must be equal, which implies the two relations given.

Answer 2 (a) [5 marks] Since $\mathbf{E_{sc}}$ and $\mathbf{B_{sc}}$ are perpendicular, and similarly for the incident electric and magnetic fields, one has $|\mathbf{S_{sc}}| = |\mathbf{E_{sc}}|^2/c\mu_0$ and similarily for the incident flux. The time averaging factors cancel in the ratio and the result follows.

(b) [5 marks] The scatterer at $\mathbf{x}_j$ experiences the incident field with a phase factor differing from that at the origin by $e^{i\mathbf{k}_0\cdot\mathbf{x}_j}$. Its response will therefore also acquire this phase factor. Likewise the phase at the detector of the component scattered by this scatterer acquires a further factor $e^{-ik\mathbf{n}\cdot\mathbf{x}_j}$compared with what would have been received from a scatterer at the origin. So the phase of the contribution to $\mathbf{E}_{\text{SC}}$ is modified by an overall factor $e^{i\mathbf{q}\cdot\mathbf{x}_j}$, so that the electric component of the scattered field is $\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}$ times what was the case for a single scatterer at the origin. Since the differential cross-section involves the square modulus of this, the result is as given, namely to multiply the result for a single scatterer by the structure factor.

(c) [5 marks] For N scatterers, the sum gives directly that $\mathcal{F}(0) = N^2$. For a large number of randomly-distributed scatterers, the phases of off-diagonal terms in the sum (obtained from expanding out the modulus squared) will cancel except close to the forward direction, provided that $|\mathbf{q}|a >> 1$. Then $\mathcal{F}(\mathbf{q}) \simeq N$.

(d) [5 marks] In a crystal, there are peaks in the structure function around $qa = 0, 2\pi, 4\pi, ...$, ie when the Bragg condition is satisfied, and then $\mathcal{F} = N^2$. The number of peaks is limited by the maximum value which qa can attain, $qa \leq 2ka$, so that at long wavelengths only the forward peak occurs. This has a width determined by $q \leq 2\pi/Na$, corresponding to scattering angles less than or of order λ/L, where L is the linear size of the crystal. (In each direction one finds $\mathcal{F}(\mathbf{q}) = \frac{\sin^2(Nqa/2)}{\sin^2(qa/2)}$; this formula is not required for full marks.)

Answer 3 (i) [6 marks] These follow from *div curl* $= 0 =$ *curl grad* and manipulations of the equations.

(ii) [2 marks] Straightforward calculation.

(iii) [4 marks] This follows as the partial derivatives commute. The gauge transformations on A_μ are

$$A_\mu \to A_\mu - \partial_\mu \Lambda$$

which leave $F_{\mu\nu}$ invariant as the derivatives commute again.

(iv) [8 marks] Write

$$\mathbf{J}(\mathbf{x}) = \int \delta^3(\mathbf{x}' - \mathbf{x})\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}' = \nabla^2 \int \frac{-1}{4\pi} \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}$$

$$= -\frac{1}{4\pi}\nabla\nabla \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' + \frac{1}{4\pi}\nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' =: \mathbf{J}_l + \mathbf{J}_t$$

with $\nabla \times \mathbf{J}_l = 0$ and $\nabla \cdot \mathbf{J}_t = 0$, so that these fields are longitudinal and transverse respectively. Now $\nabla^2\Phi = -\frac{1}{\epsilon_0}\rho$ in the Coulomb gauge, so that

$$\Phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}'$$

whence

$$\frac{1}{c^2}\,\nabla\dot{\Phi} = \frac{1}{4\pi\epsilon_0 c^2}\nabla \int \frac{\dot{\rho}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' = -\frac{\mu_0}{4\pi}\nabla \int \nabla' \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}\, d^3\mathbf{x}' = \mu_0 \mathbf{J}_l$$

(as $\nabla \cdot \mathbf{J} + \dot{\rho} = 0$). Thus

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \frac{1}{c^2}\nabla\dot{\Phi} = -\mu_0\mathbf{J}_t.$$

Answer 4 (i) [6 marks] The first result follows directly if the student knows the definition of the Poynting vector $\mathbf{S}_{\text{far}} = \frac{1}{\mu_0}\mathbf{E}_{\text{far}} \times \mathbf{B}_{\text{far}}$. The second requires some use of the vector identities given in the supplied formula sheet.

(ii) [3 marks] This follows directly from the formula above.

(iii) [6 marks] Specialising to the case of motion in a circle, where

$$\left|\frac{d\mathbf{p}}{d\tau}\right| = \left|\gamma\frac{d\mathbf{p}}{dt}\right| = \gamma\omega|\mathbf{p}|,$$

if the energy loss per revolution is small,

$$\frac{1}{c}\frac{dE}{d\tau} << \left|\frac{d\mathbf{p}}{d\tau}\right|,$$

we have

$$\begin{aligned} P &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\frac{1}{m^2}\gamma^2\omega^2|\mathbf{p}|^2 \\ &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c}\frac{1}{m^2}\gamma^2\omega^2\gamma^2\beta^2m^2 \\ &= \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}c\beta^4\gamma^4\frac{1}{\rho^2} \end{aligned}$$

using that the radius $\rho = c\beta/\omega$ for motion in a circle.

(iv) [5 marks] The energy lost in a single revolution is $\Delta E = P\times$ the time for a single revolution,

$$\begin{aligned} \Delta E &= P\frac{2\pi}{\omega} \\ &= P\,2\pi\,\frac{\rho}{c\beta} \\ &= \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{\rho}\beta^3\gamma^4 \\ &= \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0}\beta^3\left(\frac{E}{mc^2}\right)^4\frac{1}{\rho}. \end{aligned}$$

Answer 5

(i) [6 marks] This follows from $\partial_\mu \frac{\partial L}{\partial \partial_\mu A_\nu} = \frac{\partial L}{\partial A_\nu}$, or equivalently $\delta L = -(1/\mu_0) F^{\mu\nu} \partial_\mu \delta A_\nu - j^\mu \delta A_\mu$ and integration by parts.

(ii) [3 marks] The divergence of the left-hand side of the equation of motion vanishes as the derivatives commute, and hence the current is conserved.

(iii) [4 marks] The field strength is gauge invariant and the variation of the current term gives $\Lambda \partial^\mu j_\mu$ up to a total derivative.

(iv) [7 marks] The first part is immediate using the definition of the field strength tensor. The second part follows from the properties of the delta function. For the final part, intuitively the Green function is the inverse of the d'Alembertian, and in momentum space this is just $-1/k^2$ - in equations, if the Fourier transform of $D(x - x')$ is $-1/k^2$, then

$$\Box_x D(x-x') \sim \int d^4k(-1/k^2)\Box_x e^{-ik\cdot(x-x')} \sim \int d^4k e^{-ik\cdot(x-x')} \sim \delta^4(x-x'),$$

as required.