MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date: ??? May 2009

Time: 10:00?

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

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YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

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1. Maxwell's equations in linear media are

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

(i) Consider a region of space V bounded by a closed surface S, and also let C be a closed contour in space with an open surface S' spanning the contour. Explaining the notation used, derive from the above equations the integral forms

$$\int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV, \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{S'} (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S}'$$

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0, \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_{S'} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}'.$$ [**6 marks**]

(ii) Consider two regions, labelled by $i = 1, 2$, containing different linear media, which meet at an infinite two-dimensional boundary, with unit normal $\mathbf{n}$ to the boundary. Let $\mathbf{E}_i, \mathbf{D}_i, \mathbf{B}_i, \mathbf{H}_i$ for $i = 1, 2$ label the electromagnetic fields in the two regions.

Using a suitable small, shallow cylinder, straddling the boundary between the two regions, with surface charge density σ, derive the boundary conditions

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma, \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0,$$

from two of the integral equations above.

Now considering a suitable small rectangle straddling the boundary, with current density $\mathbf{K}$ on the surface of the rectangle, derive the further boundary conditions

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}, \quad \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0,$$

[**8 marks**]

(iii) Consider incident, refracted and reflected waves at this matter interface, with

$$\mathbf{E}_{inc} = \mathbf{E}_0 e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})}, \ \mathbf{E}_{refr} = \mathbf{E}'_0 e^{-i(\omega t - \mathbf{k}' \cdot \mathbf{x})}, \ \mathbf{E}_{refl} = \mathbf{E}''_0 e^{-i(\omega t - \mathbf{k}'' \cdot \mathbf{x})}.$$

Assume that the matter interface is at $z = 0$, and that the incident wave has electric field parallel to the $z - x$ plane. Let the angles of incidence, refraction and reflection be $\theta, \theta', \theta''$ respectively. Show that the boundary conditions on the fields $\mathbf{E}$ at the interface imply that

$$-E_0 \cos\theta e^{ikx\sin\theta} + E''_0 \cos\theta'' e^{ikx\sin\theta''} = -E'_0 \cos\theta' e^{ik'x\sin\theta'}$$

must be true for all x. Show that this implies that $\theta = \theta''$ (the law of reflection), and $k \sin\theta = k' \sin\theta'$ (Snell's law). [**6 marks**]

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2. In the dipole approximation for a scattering centre at the origin, the electric and magnetic fields for the scattered radiation are given in standard notation by

$$\mathbf{E}_{\text{SC}} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c],$$

$$\mathbf{B}_{\text{SC}} = \mathbf{n}\times\mathbf{E}_{\text{SC}}/c;$$

where $\mathbf{p}$ and $\mathbf{m}$ are the induced electric dipole and magnetic dipole moments of the scatterer. If the incident wave is a plane wave given by

$$\mathbf{E}_{\text{in}} = \mathbf{E}_0 e^{i\mathbf{k}_0\cdot\mathbf{x}},$$

$$\mathbf{B}_{\text{in}} = \mathbf{n}_0\times\mathbf{E}_{\text{in}}/c,$$

with $\mathbf{k}_0 = k\mathbf{n}_0$, the differential scattering cross-section may be written as

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \frac{r^2\langle|\mathbf{S}_{\text{SC}}|\rangle}{\langle|\mathbf{S}_{\text{in}}|\rangle},$$

where $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ is the Poynting flux vector and the notation $\langle\cdots\rangle$ indicates time-averaging.

(a) Show that this reduces to

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2\frac{1}{E_0^2}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c]^2.$$ **[5 marks]**

(b) Now consider a collection of identical dipole scattering centres, located at the points $\mathbf{x}_j$. Show that the effect is to multiply the cross-section for a single scatterer by the structure factor

$$\mathcal{F}(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2,$$ **[5 marks]**

where $\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n})$.

(c) Show that for N scatterers $\mathcal{F}(0) = N^2$, and find an approximation for $\mathcal{F}(\mathbf{q})$ for $N >> 1$ scatterers distributed at random, with a typical distance apart a, for $|\mathbf{q}|a >> 1$. **[5 marks]**

(d) Explain what happens if the scatterers are spaced regularly, as for example in a crystal. **[5 marks]**

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3. Consider the Maxwell equations in a vacuum with sources -

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}.$$

(i) Show that the first two of these equations may be solved by introducing the potentials $\mathbf{A}$ and Φ, and writing

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Show that the other two Maxwell equations then become

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho,$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) = -\mu_0 \mathbf{J}.$$

[6 marks]

(ii) Show that the definitions of the potentials are unchanged if we make the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda, \quad \Phi \to \Phi - \frac{\partial \Lambda}{\partial t}$$

for any function Λ. **[2 marks]**

(iii) In Lorentz covariant notation, Maxwell's equations above may be written

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad \partial^\mu F_{\mu\nu} = -\mu_0 j_\nu.$$

Show that the first of these equations is solved by writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Write down the gauge transformations on A_μ and show that they leave $F_{\mu\nu}$ invariant. **[4 marks]**

(iv) Consider Maxwell's equations in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Show that the equation for $\mathbf{A}$ can be written

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_t,$$

where $\mathbf{J}_t$ is transverse ($\nabla \cdot \mathbf{J}_t = 0$). You may use the result that $\nabla^2 \frac{1}{|\mathbf{x}' - \mathbf{x}|} = -4\pi\delta^3(\mathbf{x}' - \mathbf{x})$ and the identities $\nabla^2 \mathbf{J} = \nabla\nabla \cdot \mathbf{J} - \nabla \times \nabla \times \mathbf{J}$, and $\mathbf{J}(\mathbf{x}) = \int \delta^3(\mathbf{x}' - \mathbf{x})\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$ for any vector field $\mathbf{J}$.

[8 marks]

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4. In the far region, for non-relativistic motion $\boldsymbol{\beta} << 1$, in standard notation the fields produced by a moving charged particle are given by

$$\mathbf{E}_{\text{far}} = \frac{q}{4\pi\epsilon_0}\frac{1}{cR}[\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})]$$
$$\mathbf{B}_{\text{far}} = \frac{1}{c}\mathbf{n}\times\mathbf{E}_{\text{far}}$$

(i) Show that the Poynting vector is given by

$$\mathbf{S}_{\text{far}} = \frac{1}{\mu_0 c}|\mathbf{E}_{\text{far}}|^2\mathbf{n},$$

and hence that the power radiated per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{c^2}[\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})]^2 = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}|\dot{\mathbf{u}}|^2\sin^2\theta,$$

where θ is the angle between the direction $\mathbf{n}$ of the field point and the instantaneous acceleration $\dot{\mathbf{u}}$ of the particle.

[6 marks]

(ii) Using $\int d\Omega\,\sin^2\theta = 8\pi/3$ derive the Larmor formula for the total instantaneous power radiated:

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\frac{1}{m^2}\left(\frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}\right),$$

where $\mathbf{p} = m\mathbf{u}$ is the momentum of the particle. **[3 marks]**

(iii) For relativistic motion, the total power radiated is given by the Larmor formula, with the replacement $\frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt} \to \frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}$. Consider periodic motion in a circle of radius $\rho = c\beta/\omega$, with frequency ω. If the energy loss per period is small, ie

$$\frac{1}{c}\frac{dE}{d\tau} << \left|\frac{d\mathbf{p}}{d\tau}\right|,$$

show that

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\frac{1}{m^2}\gamma^2\omega^2|\mathbf{p}|^2 = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}c\beta^4\gamma^4\frac{1}{\rho^2}$$

[6 marks]

(iv) Using the result in (iii) above, show that the energy loss per revolution is given by the synchroton radiation formula

$$\Delta E = \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0}\beta^3\left(\frac{E}{mc^2}\right)^4\frac{1}{\rho}.$$

[5 marks]

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5. (i) The Lagrangian density for the electromagnetic field is

$$L = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu.$$

Show that this Lagrangian implies that the equations of motion of the field A_μ are

$$\partial_\mu F^{\mu\nu} = -\mu_0 j^\nu.$$

[6 marks]

(ii) Show that these equations of motion automatically imply conservation of the current j^ν. **[3 marks]**

(iii) Show that the Lagrangian in part (i) above is invariant under the gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda$, for any function Λ, if the current is conserved (you may set to zero any total derivatives). **[4 marks]**

(iv) Show that the equations of motion in the Lorentz gauge $\partial_\mu A^\mu = 0$ are

$$\Box A^\mu = -\mu_0 j^\mu,$$

where $\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$.

Show that a solution of this equation is given in standard notation by

$$A^\mu(x) = -\mu_0 \int D(x - x') j^\mu(x') d^4x',$$

if the function $D(x - x')$ satisfies

$$\Box_x D(x - x') = \delta^4(x - x').$$

Sketch an argument as to why the four-dimensional Fourier transform of $D(x)$ is given by

$$\tilde{D}(k) = -\frac{1}{k^2}.$$

[7 marks]

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Formula Sheet

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c},\\
\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a},\\
\nabla\times(\psi\mathbf{a}) &= (\nabla\psi)\times\mathbf{a}+\psi(\nabla\times\mathbf{a}),\\
\nabla\times(\nabla\times\mathbf{a}) &= \nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a},\\
\nabla\big(\psi(r)\big) &= \mathbf{n}\psi'(r).
\end{aligned}$$

Maxwell's equations:

$$\begin{aligned}
\nabla\cdot\mathbf{B}=0, \qquad & \nabla\times\mathbf{E}=-\tfrac{\partial\mathbf{B}}{\partial t};\\
\nabla\cdot\mathbf{D}=\rho, \qquad & \nabla\times\mathbf{H}=\mathbf{J}+\tfrac{\partial\mathbf{D}}{\partial t}.
\end{aligned}$$

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

$$\nabla\cdot\mathbf{J}+\dot{\rho}=0.$$

For linear isotropic media:

$$\mathbf{D}=\epsilon\mathbf{E}=\epsilon_0\mathbf{E}+\mathbf{P}, \qquad \mathbf{H}=\frac{1}{\mu}\mathbf{B}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}.$$

$$c^2d\tau^2=c^2dt^2-dx^2-dy^2-dz^2=dx^\alpha\eta_{\alpha\beta}dx^\beta.$$

$$\eta_{\alpha\beta}=\begin{cases}+1 & \text{if } \alpha=\beta=0\\ -1 & \text{if } \alpha=\beta=1,2,3\\ 0 & \text{if } \alpha\neq\beta\end{cases}$$

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\nabla\right), \qquad \partial^\mu=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right).$$

$$\partial_\alpha F^{\alpha\beta}=\partial_\alpha\partial^\alpha A^\beta-\partial^\beta\partial_\alpha A^\alpha=\mu_0 j^\beta; \qquad F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0.$$

$$\|F^{\alpha\beta}\|=\begin{pmatrix}0 & -E^1/c & -E^2/c & -E^3/c\\ E^1/c & 0 & -B^3 & B^2\\ E^2/c & B^3 & 0 & -B^1\\ E^3/c & -B^2 & B^1 & 0\end{pmatrix}.$$

End of Examination Paper **Prof WJ Spence**