MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date: May 2008

Time: hh:mm

Course Organiser: Prof WJ Spence

Deputy CO: Dr O Soloviev

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

This page should be blank

1. Maxwell's equations in linear media are

$$\nabla\cdot\mathbf{D}=\rho,\quad \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t},\quad \nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}.$$

(i) Consider a region of space V bounded by a closed surface S, and also let C be a closed contour in space with an open surface S' spanning the contour. Explaining the notation used, derive from the above equations the integral forms

$$\int_S \mathbf{D}\cdot d\mathbf{S}=\int_V \rho\, dV,\quad \oint_C \mathbf{H}\cdot d\mathbf{l}=\int_{S'}(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t})\cdot d\mathbf{S}'$$

$$\int_S \mathbf{B}\cdot d\mathbf{S}=0,\quad \oint_C \mathbf{E}\cdot d\mathbf{l}=-\int_{S'}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}'.$$ [6 marks]

(ii) Consider two regions, labelled by $i=1,2$, containing different linear media, which meet at an infinite two-dimensional boundary, with unit normal $\mathbf{n}$ to the boundary. Let $\mathbf{E}_i, \mathbf{D}_i, \mathbf{B}_i, \mathbf{H}_i$ for $i=1,2$ label the electromagnetic fields in the two regions.

Using a suitable small, shallow cylinder, straddling the boundary between the two regions, with surface charge density σ, derive the boundary conditions

$$(\mathbf{D}_2-\mathbf{D}_1)\cdot\mathbf{n}=\sigma,\quad (\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0,$$

from two of the integral equations above.

Now considering a suitable small rectangle straddling the boundary, with current density $\mathbf{K}$ on the surface of the rectangle, derive the further boundary conditions

$$\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K},\quad \mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1)=0,$$

[8 marks]

(iii) Consider incident, refracted and reflected waves at this matter interface, with

$$\mathbf{E}_{inc}=\mathbf{E}_0 e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})},\ \mathbf{E}_{refr}=\mathbf{E}_0' e^{-i(\omega t-\mathbf{k}'\cdot\mathbf{x})},\ \mathbf{E}_{refl}=\mathbf{E}_0'' e^{-i(\omega t-\mathbf{k}''\cdot\mathbf{x})}.$$

Assume that the matter interface is at $z=0$, and that the incident wave has electric field parallel to the $z-x$ plane. Let the angles of incidence, refraction and reflection be $\theta, \theta', \theta''$ respectively. Show that the boundary conditions on the fields $\mathbf{E}$ at the interface imply that

$$-E_0\cos\theta e^{ikx\sin\theta}+E_0''\cos\theta'' e^{ikx\sin\theta''}=-E_0'\cos\theta' e^{ik'x\sin\theta'}$$

must be true for all x. Show that this implies that $\theta=\theta''$ (the law of reflection), and $k\sin\theta=k'\sin\theta'$ (Snell's law). [6 marks]

please turn to the next page

2. For an oscillating electric dipole with strength $\mathbf{p}$, oscillating in time as $e^{-i\omega t}$, the vector potential is given by

$$\mathbf{A}^{e.d.}(\mathbf{r},t) = -\frac{1}{4\pi\epsilon}\frac{e^{ikr}}{r}\frac{ik}{c}\;\mathbf{p}\;e^{-i\omega t},$$

where $\mathbf{r} = (x, y, z)$, $r = \sqrt{x^2+y^2+z^2}$ and $k = \omega/c$.

(i) Show that in the far zone, where $kr >> 1$, this results in the magnetic field

$$\mathbf{B}^{e.d.}(\mathbf{p}) = \frac{1}{4\pi\epsilon}\frac{k^2}{c}\;\mathbf{n}\times\mathbf{p}\;\frac{e^{ikr}}{r}\;e^{-i\omega t}$$

where $\mathbf{n} = \frac{1}{r}\mathbf{r}$. [4 marks]

(ii) For the electric field $\mathbf{E}^{e.d.}(\mathbf{p})$, use the source-free Maxwell equation $\dot{\mathbf{E}} = c^2\nabla\times\mathbf{B}$ and the fact that the time dependence of the fields is $e^{-i\omega t}$ to deduce that

$$\mathbf{E}^{e.d.}(\mathbf{p}) = \frac{ic}{k}\;\nabla\times\mathbf{B}^{e.d.}(\mathbf{p})\;e^{-i\omega t}$$

and hence that in the far zone

$$\mathbf{E}^{e.d.}(\mathbf{p}) = c\;\mathbf{B}^{e.d.}(\mathbf{p})\times\mathbf{n}\;.$$

[5 marks]

(iii) The vector potential for an oscillating *magnetic* dipole is given by

$$\mathbf{A}^{m.d.}(\mathbf{r},t) = ik\;\frac{\mu_0}{4\pi}\;\frac{e^{ikr}}{r}\;\mathbf{n}\times\mathbf{m}\;e^{-i\omega t}.$$

Show that this is proportional to the magnetic field for the electric dipole, with $\mathbf{p}$ replaced by $\mathbf{m}$:

$$\mathbf{A}^{m.d.} = \frac{i}{kc}\;\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m}).$$

Thus prove that the electric and magnetic fields for a magnetic dipole are given by

$$\mathbf{B}^{m.d.}(\mathbf{m}) = \frac{1}{c^2}\;\mathbf{E}^{e.d.}(\mathbf{p}\to\mathbf{m}),$$

$$\mathbf{E}^{m.d.}(\mathbf{m}) = -\;\mathbf{B}^{e.d.}(\mathbf{p}\to\mathbf{m}).$$

[8 marks]

(iv) How are the polarisation vectors, the directions of the magnetic fields, and the directions of the radiation $\mathbf{n}$ oriented with respect to each other in the two cases of electric and magnetic dipole radiation in the far zone ?

[3 marks]

please turn to the next page

3. (i) Show in the Lorentz gauge ($\partial^\mu A_\mu = 0$), with $A^\mu = (\frac{1}{c}\Phi, \mathbf{A})$ and $j^\mu = (c\rho, \mathbf{J})$, that the Maxwell equation $\partial^\mu F_{\mu\nu} = \mu_0 j_\nu$ reduces to

$$\partial^\mu \partial_\mu \mathbf{A} = \mu_\mathbf{0} \mathbf{J}, \qquad \partial^\mu \partial_\mu \mathbf{\Phi} = \frac{\mathbf{1}}{\epsilon_\mathbf{0}} \rho.$$

[3 marks]

(ii) Integrate the equation for $\mathbf{A}$ above with $\int_{-\infty}^{\infty} e^{-i\omega t}$ to obtain the Fourier transformed equation

$$(\nabla^2 + k^2)\mathbf{A}(\mathbf{x}, \omega) = -\mu_\mathbf{0} \mathbf{J}(\mathbf{x}, \omega), \qquad (\mathbf{1})$$

with $k^2 = \omega^2/c^2$. [4 marks]

(iii) Suppose that there exists a Green function $G_k(\mathbf{x}, \mathbf{x}')$, satisfying

$$(\nabla^2 + k^2) G_k(\mathbf{x}, \mathbf{x}') = -\mathbf{4}\pi\delta^\mathbf{3}(\mathbf{x} - \mathbf{x}'). \quad (\mathbf{2})$$

Show that

$$\mathbf{A}(\mathbf{x}, \omega) = \frac{\mu_\mathbf{0}}{\mathbf{4}\pi} \int \mathbf{G_k}(\mathbf{x}, \mathbf{x}') \mathbf{J}(\mathbf{x}', \omega) \mathbf{d^3 x'}$$

solves equation (1) above. [3 marks]

(iv) Give an argument why $G_k(\mathbf{x}, \mathbf{x}')$ must be purely a function of $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$. Show that in this case equation (2) becomes

$$\frac{1}{r} \frac{d^2}{dr^2} (r G_k(r)) + k^2 G_k(r) = -4\pi \delta^3(\mathbf{r})$$

and hence that when $r \neq 0$, $G_k(r)$ is given by

$$G_k(r) = \frac{1}{r}(A e^{ikr} + B e^{-ikr}), \quad (3)$$

for some constants A, B. [5 marks]

(v) A solution of Poisson's equation $\nabla^2 \phi = -\frac{1}{\epsilon_0}\rho$ is $\phi = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$. Use this fact to show that when $r \to 0$, (3) above remains a solution of equation (2) if

$$A + B = 1.$$

[5 marks]

please turn to the next page

4. Consider the Maxwell equations in a vacuum with sources -

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}.$$

(i) Show that the first two of these equations may be solved by introducing the potentials $\mathbf{A}$ and Φ, and writing

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Show that the other two Maxwell equations then become

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho,$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}) = -\mu_0 \mathbf{J}.$$

[6 marks]

(ii) Show that the definitions of the potentials are unchanged if we make the gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\Lambda, \quad \Phi \to \Phi - \frac{\partial \Lambda}{\partial t}$$

for any function Λ. [2 marks]

(iii) In Lorentz covariant notation, Maxwell's equations above may be written

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad \partial^\mu F_{\mu\nu} = -\mu_0 j_\nu.$$

Show that the first of these equations is solved by writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Write down the gauge transformations on A_μ and show that they leave $F_{\mu\nu}$ invariant. [4 marks]

(iv) Consider Maxwell's equations in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Show that the equation for $\mathbf{A}$ can be written

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_t,$$

where $\mathbf{J}_t$ is transverse ($\nabla \cdot \mathbf{J}_t = 0$). You may use the result that $\nabla^2 \frac{1}{|\mathbf{x}'-\mathbf{x}|} = -4\pi\delta^3(\mathbf{x}'-\mathbf{x})$ and the identities $\nabla^2\mathbf{J} = \nabla\nabla\cdot\mathbf{J} - \nabla\times\nabla\times\mathbf{J}$, and $\mathbf{J}(\mathbf{x}) = \int \delta^3(\mathbf{x}'-\mathbf{x})\mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$ for any vector field $\mathbf{J}$.

[8 marks]

please turn to the next page

5. The electric and magnetic fields generated by a charged particle moving with velocity $c\beta$ and acceleration $c\dot{\beta}$ are given by the Lienard-Wiechert expressions

$$\mathbf{B} = \frac{1}{c}[\mathbf{n} \times \mathbf{E}]_{\text{ret}},$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n}-\beta}{\gamma_u^2 R^2(1-\beta\cdot\mathbf{n})^3}\right]_{\text{ret}} + \frac{q}{4\pi\epsilon_0}\frac{1}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\beta)\times\dot{\beta}]}{(1-\beta\cdot\mathbf{n})^3 R}\right]_{\text{ret}},$$

where $\mathbf{n}$ is the unit vector which points from the point on the particle trajectory to the field point $\mathbf{x}$, with $\mathbf{x} = \mathbf{r}(\tau_0) = \mathbf{n}R$, and the retarded time is $t_{ret} = t - R/c$.

(i) Show that for the case when the acceleration is parallel to the velocity, the electric field far from the charge is given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{c}\left[\frac{\mathbf{n}\times[\mathbf{n}\times\dot{\beta}]}{(1-\beta\cdot\mathbf{n})^3 R}\right]_{\text{ret}}.$$

[2 marks]

(ii) Show that in this case, the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$, far from the charge is

$$\mathbf{S} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\left[\frac{\dot{\beta}\sin\theta}{(1-\beta\cdot\mathbf{n})^3 R}\right]^2_{\text{ret}}\mathbf{n},$$

where θ is the angle between $\mathbf{n}$ and the common direction of the velocity and acceleration of the particle. [4 marks]

(iii) Show that the power radiated per unit solid angle is given by

$$\frac{dP(t')}{d\Omega} = R^2\mathbf{n}\cdot\mathbf{S}\frac{dt}{dt'}.$$

and hence equals

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c}\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}.$$

[6 marks]

(iv) For non-relativistic motion, deduce from this the Larmor formula

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4\pi c^3}\dot{u}^2\sin^2\theta.$$

[2 marks]

(v) Without making the non-relativistic approximation, show that the maximum intensity of radiation is observed at the angle

$$\theta_{\text{max}} = \cos^{-1}\left[\frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)\right].$$

[6 marks]

please turn to the next page

Formula Sheet

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c},\\
\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a},\\
\nabla\times(\psi\mathbf{a}) &= (\nabla\psi)\times\mathbf{a}+\psi(\nabla\times\mathbf{a}),\\
\nabla\times(\nabla\times\mathbf{a}) &= \nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a},\\
\nabla\big(\psi(r)\big) &= \mathbf{n}\psi'(r).
\end{aligned}$$

Maxwell's equations:

$$\begin{aligned}
\nabla\cdot\mathbf{B}&=0, &\qquad \nabla\times\mathbf{E}&=-\tfrac{\partial\mathbf{B}}{\partial t};\\
\nabla\cdot\mathbf{D}&=\rho, &\qquad \nabla\times\mathbf{H}&=\mathbf{J}+\tfrac{\partial\mathbf{D}}{\partial t}.
\end{aligned}$$

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

$$\nabla\cdot\mathbf{J}+\dot\rho=0.$$

For linear isotropic media:

$$\mathbf{D}=\epsilon\mathbf{E}=\epsilon_0\mathbf{E}+\mathbf{P},\qquad \mathbf{H}=\frac{1}{\mu}\mathbf{B}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}.$$

$$c^2d\tau^2=c^2dt^2-dx^2-dy^2-dz^2=dx^\alpha\eta_{\alpha\beta}dx^\beta.$$

$$\eta_{\alpha\beta}=\begin{cases}+1 & \text{if } \alpha=\beta=0\\ -1 & \text{if } \alpha=\beta=1,2,3\\ 0 & \text{if } \alpha\neq\beta\end{cases}$$

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\nabla\right),\qquad \partial^\mu=\left(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\right).$$

$$\partial_\alpha F^{\alpha\beta}=\partial_\alpha\partial^\alpha A^\beta-\partial^\beta\partial_\alpha A^\alpha=\mu_0 j^\beta;\qquad F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0.$$

$$\|F^{\alpha\beta}\|=\begin{pmatrix}0 & -E^1/c & -E^2/c & -E^3/c\\ E^1/c & 0 & -B^3 & B^2\\ E^2/c & B^3 & 0 & -B^1\\ E^3/c & -B^2 & B^1 & 0\end{pmatrix}.$$

In spherical coordinates (r,θ,ϕ), for a scalar field $G(r,\theta,\phi)$,

$$\nabla^2G=\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2G}{\partial\phi^2}.$$

A solution of Poisson's equation $\nabla^2\phi=-\frac{1}{\epsilon_0}\rho$ is $\phi=\frac{1}{4\pi\epsilon}\int\frac{\rho(\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'$.

End of Examination Paper **Prof WJ Spence**