Queen Mary and Westfield College

UNIVERSITY OF LONDON

MSc in Astrophysics

ASTM111/MAS414 Solar System Dynamics

12 May 2000   10.00 am – 1.00 pm

Time Allowed: 3 hours

FULL marks may be obtained for a complete solution to THREE questions. ALL questions carry equal marks, but it is suggested that candidates from 1999-2000 consider Questions 1–4 and resitting candidates consider Questions 1,2,5,6.

Calculators may be used in this examination but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

You are reminded of the following, which you may use without proof.

In all questions: $M_\odot$ is the mass of the Sun, r is the heliocentric distance and G the gravitational constant.

You may assume that the equation of an ellipse is

$$\frac{a(1-e^2)}{r} = 1 + e\cos f$$

and that

$$M = nt = E - e\sin E,$$

where n is the mean motion.

The following rounded numerical values, all in S.I. Units may be assumed throughout the paper.

$$G = 7 \times 10^{-11}, M_\odot = 2 \times 10^{30}.$$

You may also assume that 1 year is 365.25 days or 3×10^7 seconds and that 1 AU is 1.5×10^{11} m.

continued overleaf ...

1. (a) With the aid of diagrams, define the *true anomaly* f, the *eccentric anomaly* E and the *argument of perihelion* ω of a body moving on an elliptic orbit. Using geometric considerations, deduce that

$$r\cos f = a(\cos E - e).$$

where r is the heliocentric distance, a the semi-major axis and e the eccentricity of the orbit. Deduce that $r = a(1 - e\cos E)$.

(b) A comet is moving about the Sun on an elliptic orbit of semi-major axis a and eccentricity e. Its speed at perihelion is v while at aphelion it is V. Use conservation of angular momentum to obtain the ratio v/V in terms of e.

Write down an equation representing the conservation of energy between perihelion and aphelion. (You may assume that the gravitational energy of a body of mass M at a heliocentric distance r is $-M\mu/r$, where $\mu = GM_\odot$.) Hence, find v and deduce that the energy per unit mass of the comet is $-\mu/(2a)$.

(c) Comet Hale-Bopp has the following approximate orbital parameters

$$\begin{array}{lll} q = 0.92 & e = 0.995 & i = 89.^\circ 43 \\ \Omega = 282.^\circ 5 & \omega = 130.^\circ 6 & P = 2500 \text{ years} \end{array}$$

where q is the perihelion distance and e is the eccentricy. Calculate the speed of the comet at perihelion in km s^{-1} and the ratio of the speeds at perihelion and aphelion. Calculate also the two values for the distance of the comet from the Sun when it crosses the ecliptic.

(d) Calculate the time taken by Comet Hale-Bopp to cover the short arc of the orbit between the node nearest Jupiter and perihelion.

2. Consider a Cartesian frame of reference (X, Y, Z) with the X-axis along the Sun-Jupiter line, the Y-axis parallel to the velocity vector of Jupiter and the Z-axis making a right-handed orthogonal set. The origin is at the mass-centre of the Sun-Jupiter system. Assume that the Sun-Jupiter distance, a_J, remains constant and that the reference frame rotates with a constant angular velocity Ω, keeping Jupiter on the X-axis. Let $M_\odot$ and M_J denote the masses of the Sun and Jupiter respectively.

(a) Show that

$$\Omega^2 = \frac{G(M_\odot + M_J)}{a_J^3}.$$

(b) A position of dynamical equilibrium exists on the line joining $M_\odot$ and M_J at a distance x from M_J. Prove that x is given by

$$M_\odot x^2\left(a_J^3 - (a_J - x)^3\right) = M_J(a_J - x)^2(a_J^3 - x^3)$$

Given that $x \ll a_J$, and $M_\odot \gg M_J$ deduce that

$$3M_\odot x^3 = M_j a_J^3$$

(c) Show that conservation of energy of angular momentum of a particle of negligible mass gives

$$\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 - \Omega^2(X^2 + Y^2) - 2GM_\odot/r_1 - 2GM_J/r_2 = \text{Constant}.$$

(d) By assuming that M_J is small and that the orbit of such a particle is an ellipse about the Sun, with semi-major axis a, eccentricity e and inclination i, show that Tisserand's invariant is

$$T = \frac{a_J}{a} + 2\cos i \sqrt{\frac{a(1-e^2)}{a_J}}.$$

(e) Comet Brooks 2 passed close to Jupiter in 1889 and in consequence its orbit changed significantly. The current (that is post 1889) orbit is given by $a = 3.7\,\mathrm{AU}$, $e = 0.5$ and $i = 6^\circ$. The previous orbit is not well known but the perihelion must have been close to Jupiter and the period must be in excess of about 50 years, otherwise it would have been observed at other apparitions. Assuming that for this orbit $q = 5.0\,\mathrm{AU}$ and $P = 64$ years, find possible values for i given that Tisserand's invariant remained valid.

You may take $a_J = 5.2\,\mathrm{AU}$.

3. Consider a system with two degrees of freedom and coordinates (q_x, q_y) and momenta (p_x, p_y), and Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(q_x, q_y, t).$$

(a) Now the potential V is rotating, i.e., $V = U(q'_x, q'_y)$ where

$$\begin{pmatrix} q'_x \\ q'_y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

Using the transformation formula

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}, \quad \mathbf{Q} = \frac{\partial S}{\partial \mathbf{P}}, \quad F = H + \frac{\partial S}{\partial t}$$

with generating function

$$S = \begin{pmatrix} P_x & P_y \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix},$$

(i) work out (Q_x, Q_y) in terms of (q_x, q_y),

(ii) work out (p_x, p_y) in terms of (P_x, P_y),

(iii) show that the transformed Hamiltonian is

$$F = \tfrac{1}{2}(P_x^2 + P_y^2) + U(Q_x, Q_y) + (Q_x P_y - Q_y P_x).$$

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