Queen Mary and Westfield College

UNIVERSITY OF LONDON

MSci by course units

MAS414 Solar System Dynamics

3 June 1999 10.00am – 1.00pm

Time Allowed: 3 hours

Calculators are NOT permitted in this examination.

You may attempt any number of questions. Full marks will be obtained by providing complete answers to about THREE questions.

continued overleaf ...

1. A planet moves under the gravitational attraction of a central star and its resulting path is an ellipse with the star at one focus. The relationship between the planet's radial distance, r from an origin O at the star and its true anomaly, f, is given in polar coordinates by,

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

where a and e are the semi-major axis and eccentricity of the orbit. The same orbital path can be described by the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

where $b = a\sqrt{1-e^2}$ is the semi-minor axis of the ellipse, (x, y) are the coordinates of the planet in a frame with origin, O', at the centre of the ellipse (midway between the two foci) and the x-axis lies along the line joining the two foci.

(a) Draw a diagram to illustrate the relationship between the polar coordinate system with origin O and the cartesian coordinate system with origin O'. Sketch a circle of radius a centred on the origin O' and use it to illustrate the relationship between f and the eccentric anomaly, E. Derive expressions for $r\cos f$ and $r\sin f$, and hence show that

$$r = a(1 - e\cos E).$$

(12 marks)

(b) Substitute the result from part (a) in the equation

$$\dot{r} = \frac{na}{r}\sqrt{a^2e^2 - (r-a)^2}$$

where n is the mean motion of the object, and hence solve it to derive Kepler's equation,

$$M = E - e\sin E$$

where $M = n(t - \tau)$ is the mean anomaly and τ is a constant. (11 marks)

(c) Derive a series solution for Kepler's equation to express E as a function of M including terms up to and including $\mathcal{O}(e^2)$. State two possible limitations to the use of such a series for numerical solutions to Kepler's equation. In solar system dynamics, why is it advantageous to express quantities as series in M rather than E? (10 marks)

2. In the planar circular restricted three-body problem the equations of motion of the test particle in the rotating frame are given by

$$\ddot{x} - 2\dot{y} = U_x \qquad \ddot{y} + 2\dot{x} = U_y$$

where

$$U = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

and $U_x = \partial U/\partial x$, $U_y = \partial U/\partial y$, $\mu_1 = m_1/(m_1+m_2)$, $\mu_2 = m_2/(m_1+m_2)$, $m_2 < m_1$. The square of the distances from the particle to the masses m_1 and m_2 are given by $r_1^2 = (x + \mu_2)^2 + y^2$, $r_2^2 = (x - \mu_1)^2 + y^2$ respectively.

(a) Show that $\dot{x}^2 + \dot{y}^2 = 2U - C$ where C is a constant and explain how the existence of this constant can be used to place bounds on the motion of the particle. Under what circumstances can this method not be used? (8 marks)

(b) Show that the equations of motion have equilibrium solutions at the points given by $x_0 = \frac{1}{2} - \mu$, $y_0 = \pm\sqrt{3}/2$. By considering a small displacement (X, Y) from (x_0, y_0), derive a set of simultaneous linearised differential equations of the form

$$(D^2 - U_{xx})X - (2D + U_{xy})Y = 0$$
$$(2D - U_{xy})X + (D^2 - U_{yy})Y = 0$$

where $D \equiv d/dt$ and the $U_{xx} = \partial^2 U/\partial x^2$, etc are the partial derivatives evaluated at (x_0, y_0). (9 marks)

(c) Derive the numerical values of U_{xx}, U_{xy} and U_{yy} at (x_0, y_0). By assuming solutions of the form $X = \alpha e^{\lambda t}$, $Y = \beta e^{\lambda t}$ where α, β and λ are constants, show that the simultaneous equations have a zero determinant provided

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu_1\mu_2 = 0$$

and hence show that the equilibrium points are linearly stable provided

$$1 - 27\mu_1\mu_2 > 0. \qquad \text{(12 marks)}$$

(d) Sketch the path in the rotating frame of a test particle with a small eccentricity placed close to the equilibrium point (x_0, y_0) in (i) the Sun–Jupiter system where $\mu_2 = 0.001$ and (ii) the Pluto–Charon system where $\mu_2 = 0.12$. Give a brief explanation for the motion you have sketched. (4 marks)

3. The asteroid Griqua orbits at the 2:1 interior mean motion resonance with the planet Jupiter with successive conjunctions of the planet and the asteroid occurring at the asteroid's perihelion. Griqua's orbit has an eccentricity $e = 0.3$ while Jupiter can be taken to be moving in a circular orbit in the same plane as Griqua.

(a) Show that the ratio of the perihelion distance, $r_{\rm p}$, to the aphelion distance, $r_{\rm a}$ can be written as

$$\frac{r_{\rm p}}{r_{\rm a}} = 1 + 2\sum_{i=1}^{\infty}(-1)^i e^i \, .$$

Sketch the orbits of Griqua and Jupiter in the inertial, non-rotating frame marking the location of the asteroid's perihelion and aphelion. Indicate possible locations of both objects ensuring that these are consistent with the nature of the resonance. (8 marks)

(b) Sketch the approximate path of Griqua in a frame rotating at a rate equal to the mean motion of Jupiter. Indicate on your plot all locations where Griqua is at the perihelion or aphelion of its orbit. (4 marks)

(c) The angular momentum per unit mass of an object orbiting the Sun is given by $h = na^2\sqrt{1-e^2}$, where n is the mean motion and a is the semi-major axis. Use this to obtain an expression for the angular velocity of the planet as a function of the true anomaly f. By calculating the angular velocity of Griqua at its aphelion and equating it to the mean motion of Jupiter, show that Griqua would be instantaneously stationary at one or more locations in the rotating frame provided e satisfies the equation

$$e^3 + 3e^2 + 7e - 3 = 0 \, . \qquad \text{(11 marks)}$$

(d) A detailed analytical study of the motion of the Griqua–Jupiter system requires an expansion of the disturbing function involving the eccentricities (e and e') and inclinations (I and I') of both objects to the second degree. This implies a total of eight cosine arguments in the expansion of the planetary disturbing function: two are associated with the 2:1 resonance and six are associated with the 4:2 resonance. Write down each cosine argument, stating clearly the angles involved and the form of the term in eccentricity and inclination associated with each argument. What is the relationship, if any, between the coefficients of the angles in the argument and the powers of eccentricity and inclination in the term? (10 marks)

4. Two planets of mass m_1 and m_2 move in coplanar, non-intersecting elliptical orbits about a homogeneous, spherical central star. The inner planet has orbital elements a_1, e_1 and ϖ_1 while the outer one has elements a_2, e_2, and ϖ_2. The secular part of the disturbing function for perturbations on m_1 due to m_2 is $R_1 = m_2 R_{12}$ while that on m_2 due to m_1 is $R_2 = m_1 R_{12}$ where

$$R_{12} = \frac{\mathcal{G}}{a_2}\left\{N_{12}\left[e_1^2 + e_2^2\right] - 2P_{12}e_1e_2\cos(\varpi_1 - \varpi_2)\right\}$$

where $\mathcal{G}$ is the universal gravitational constant and N_{12} and P_{12} are constants. When R_{12} is expressed in terms of the variables

$$h_i = e_i \sin\varpi_i \qquad k_i = e_i \cos\varpi_i$$

the equations of motion can be written

$$\dot{h}_i = +\frac{1}{n_i a_i^2}\frac{\partial R_i}{\partial k_i} \qquad \dot{k}_i = -\frac{1}{n_i a_i^2}\frac{\partial R_i}{\partial h_i}$$

where n_1 and n_2 are the mean motions of m_1 and m_2.

(a) Why is it preferable to use the elements h and k rather than e and ϖ?

(4 marks)

(b) Derive expressions for R_1 and R_2 in terms of the h_j and k_j ($j = 1, 2$) and hence show that the equations of motion can be written as the eigenvalue problem

$$\dot{h}_i = +\sum_{j=1}^{2} A_{ij}k_j \qquad \dot{k}_i = -\sum_{j=1}^{2} A_{ij}h_j$$

giving expressions for the constants A_{ij}. Write down the solutions to these equations (the Laplace-Lagrange secular solution) explaining the physical significance of the constants of integration.

(17 marks)

(c) What are the implicit assumptions in the classical Laplace-Lagrange secular solution for a system of n gravitationally interacting bodies orbiting a central object?

(6 marks)

(d) The results of a long-term numerical integration of the motion of a non-coplanar system of a system of satellites with low eccentricities and inclinations orbiting around a planet are compared with the analytical solution of the classical Laplace-Lagrange theory. The comparison reveals a large discrepancy between the eccentricity and pericentre solution derived from the numerical integration and the equivalent theoretical solution derived for two of the satellites. However, comparison of the corresponding inclination and node solution using both methods shows excellent agreement for all the satellites. Suggest a plausible explanation for these results.

(6 marks)

End of Examination

C.D. Murray