Queen Mary & Westfield College
UNIVERSITY OF LONDON

M.Sci Mathematics

MAS402 Astrophysical Fluid Dynamics

Duration: 3 hours
28th May 1999 10:00

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best **3** *questions answered will be counted.*

You may use a calculator provided that you do not make use of any programming, graph plotting or algebraic facilities it may have.

Notation

The following notation is used throughout unless otherwise stated. The pressure, density, gravitational potential and adiabatic exponents are denoted by p, ρ, ψ, Γ_1 and Γ_3 respectively. The equilibrium values of these quantities are sometimes distinguished using a zero subscript. The position vector is denoted by $\mathbf{r}$ or $\mathbf{x}$, the time by t, the velocity by $\mathbf{u}$, the surface radius of a spherical configuration by R, and the gravitational constant by G. Vectors are denoted by boldface type.

Astronomical and Physical Data

Mass of the Sun	$M_\odot$	$2.0 \times 10^{30}\,\mathrm{kg}$
Surface radius of the Sun	$R_\odot$	$7.0 \times 10^{8}\,\mathrm{m}$
Luminosity of the Sun	$L_\odot$	$3.8 \times 10^{26}\,\mathrm{J\,s^{-1}}$
Gravitational constant	G	$6.67 \times 10^{-11}\,\mathrm{kg^{-1}m^3s^{-2}}$
Speed of light in a vacuum	c	$3.0 \times 10^{8}\,\mathrm{m\,s^{-1}}$

Standard Formulae

Candidates may assume the following set of basic equations and formulae:

In spherical polar coordinates (r, θ, ϕ)

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)$$

and

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\,.$$

For $\mathbf{u} = (u_r, u_\theta, u_\phi)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

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and

$$\nabla\times\mathbf{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ u_r & ru_\theta & r\sin\theta u_\phi \end{vmatrix} .$$

The spherical harmonic $Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\exp(im\phi)$, where $P_l^{|m|}$ denotes the associated Legendre function, satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_l^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_l^m}{\partial\phi^2} + l(l+1)Y_l^m = 0 ,$$

where l is a non-negative integer and m is an integer such that $|m| \le l$. Further

$$\nabla^2(Y_l^m r^l) = 0 \qquad \nabla^2(Y_l^m r^{-l-1}) = 0 .$$

In cylindrical polar coordinates (r,ϕ,z), with $\mathbf{u} = (u_r, u_\phi, u_z)$,

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} , \quad \nabla\times\mathbf{u} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\phi & \mathbf{e}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ u_r & ru_\phi & u_z \end{vmatrix} .$$

The material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla .$$

The equation of motion for an inviscid fluid may be assumed in the form

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p - \rho\nabla\psi .$$

The continuity equation may be assumed in the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 .$$

The energy equation may be assumed in the form

$$\frac{Dp}{Dt} - \frac{\Gamma_1 p}{\rho}\frac{D\rho}{Dt} = \rho(\Gamma_3 - 1)\left(\epsilon - \frac{1}{\rho}\nabla\cdot\mathbf{F}\right) ,$$

where ϵ is the heat generated per unit mass, and $\mathbf{F}$ is the heat flux. For adiabatic motion, the right-hand side of this equation is zero.

The gravitational potential satisfies Poisson's equation, $\nabla^2\psi = 4\pi G\rho$, which may be assumed to have the solution

$$\psi(\mathbf{r},t) = -\int_V \frac{G\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V' ,$$

where the integration is taken over the fluid volume V, and $\mathrm{d}V'$ denotes the volume element $\mathrm{d}^3\mathbf{r}'$.

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1 (a) Derive the virial theorem for a gaseous mass moving under forces due to pressure and self-gravity in the form

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2K + 3\int_V p\,dV + \Psi\,,$$

where I, K and Ψ are defined to be

$$I = \int_V \rho|\mathbf{r}|^2\,dV\,, \qquad K = \frac{1}{2}\int_V \rho|\mathbf{u}|^2\,dV\,,$$

and

$$\Psi = -\frac{1}{2}G\int_V\int_V \frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,dV\,dV'\,.$$

The integrations are over the fluid volume, V, on the surface of which $p = \rho = 0$. What physically do K and Ψ represent?

(b) The gas obeys an isothermal equation of state such that $p = a^2\rho$, where a is a constant. What physically does the quantity a represent? At time $t = 0$, the fluid rotates uniformly such that

$$\mathbf{u} = \boldsymbol{\Omega}\times\mathbf{r}\,,$$

where the origin of coordinates is at the centre of mass of the fluid and $\boldsymbol{\Omega}$ is the constant angular velocity. In addition, the maximum distance between two points on the zero density surface is D. Calculate the value of $\mathrm{d}I/\mathrm{d}t$ at time $t = 0$ and show that at the same instant

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \le 2M(D^2|\boldsymbol{\Omega}|^2 + 3a^2) - \frac{GM^2}{D}\,,$$

where M is the total mass of the fluid. Give a condition that will ensure that the mass begins to collapse. Discuss the physical origin of the various terms in the above inequality for $\mathrm{d}^2I/\mathrm{d}t^2$.

(c) Describe in qualitative terms the collapse of a rotating gas cloud, and discuss whether you think the condition you derived in part (b) is a very strong condition for the collapse of a rotating cloud.

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2 An infinite layer of gas of uniform density lies between the two planes $z = \pm H$. Starting from Poisson's equation, or otherwise, find the gravitational potential $\psi(z)$ at all points. Show that in the limit in which $H \to 0$ and $2\rho H \to \Sigma_0$, the gravitational potential becomes

$$\psi = 2\pi G \Sigma_0 |z| .$$

A thin layer of gas lies in the $z = 0$ plane and has surface density $\Sigma(x,t)$. Considering only motions $(u,0,0)$ in the x-direction, and assuming that variations depend only on x, show that the equations of motion for the layer are

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial x}(\Sigma u) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\Sigma}\frac{\partial P}{\partial x} - \frac{\partial \psi}{\partial x} ,$$

where P is the integrated pressure force per unit length across the sheet.

In equilibrium, $P = P_0$, $\Sigma = \Sigma_0$ are both constants, $u = 0$ and $\psi = \psi_0(z)$. The equilibrium is now perturbed, so that perturbed quantities (which are assumed small enough for the equations to be linearized) are proportional to $\exp(ikx + i\omega t)$, where k is positive, e.g.

$$\Sigma = \Sigma_0 + \Sigma' e^{ikx+i\omega t} .$$

Show that the corresponding gravitational potential is

$$\psi = \psi_0 - 2\pi G \Sigma' k^{-1} e^{-k|z|} e^{ikx+i\omega t} .$$

Assuming that $P \propto \Sigma^{\Gamma}$, where Γ is a constant, derive the dispersion relation relating ω and k. Identify the physical origin of the various terms. Sketch the variation of ω^2 as a function of k. How does this differ qualitatively from the dispersion relation for plane perturbations of an infinite three-dimensional gas cloud?

Assuming the above analysis can be applied to a cloud in the form of a circular disk, derive the cloud's Jeans mass. What is the most rapidly growing scale of variation? How do these two quantities scale with density during the collapse if the cloud

(a) radiates heat away efficiently, and

(b) is opaque to radiation, with $\Gamma = 5/3$?

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3 A self-gravitating star of radius R has uniform density ρ. Derive expressions for the gravitational acceleration and mass $m(r)$ as functions of r, and deduce that the pressure is given by

$$p = \frac{2\pi}{3} G\rho^2 \left(R^2 - r^2\right) .$$

The star undergoes small-amplitude, radial, adiabatic oscillations. Let $\xi = u_r/r$, where $u_r e^{i\omega t}$ is the radial velocity of a fluid element from its equilibrium position. Starting from the linearized perturbed fluid equations, derive the linear adiabatic radial pulsation equation in the form

$$\frac{d^2\xi}{dx^2} + \left(\frac{4}{x} - \frac{2x}{1-x^2}\right)\frac{d\xi}{dx} + \frac{A\xi}{1-x^2} = 0 ,$$

where

$$A = \frac{3\omega^2}{2\pi G\Gamma_1 \rho} - \frac{2(3\Gamma_1 - 4)}{\Gamma_1}$$

and $x = r/R$.

The appropriate boundary conditions satisfied by ξ are that it is regular at $x = 0$ and $x = 1$. Show by direct substitution that

$$\xi_0 = a_0 \qquad \text{and} \qquad \xi_1 = a_1 + b_1 x^2$$

are both eigenfunctions for suitable constants a_0, a_1, b_1 and eigenfrequencies ω, all of which you should determine. [You should normalize the eigenfunctions so that $\xi = 1$ at $x = 1$, and you should take $\Gamma_1 = 5/3$.] Sketch the eigenfunctions as functions of x on a single diagram. Do you think these are the eigenfunctions with the two lowest eigenfrequencies? Justify your answer.

Rewrite the pulsation equation in the form

$$\frac{d}{dx}\left(\mathcal{P}\frac{d\xi}{dx}\right) + \mathcal{Q}\xi + \omega^2\mathcal{R}\xi = 0 ,$$

where $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are functions of x (independent of ω) that you should derive. Hence, show by direct evaluation that ξ_0 and ξ_1 satisfy the orthogonality condition $\int_0^1 \mathcal{R}\xi_0^*\xi_1 \, dx = 0$.

Derive a variational principle of the form

$$\omega^2 = F[\xi]$$

for radial pulsations of this star. [You are not required to *prove* that it is a variational principle.] Which eigenfrequency would the trial function

$$\xi = 2x - 1$$

be most suited for estimating? Evaluate your $F[\xi]$ for this trial function.

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4 A star with constant adiabatic exponent Γ_1 undergoes small-amplitude nonradial oscillations about a spherically symmetric equilibrium state. Show that, in the Cowling approximation, the radial velocity $u_r(r)Y_l^m(\theta,\phi)\exp(i\omega t)$ and Eulerian pressure perturbation $p'(r)Y_l^m(\theta,\phi)\exp(i\omega t)$ satisfy the equations

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(p^{-1/\Gamma_1}p'\right) = -p^{-1/\Gamma_1}\rho u_r\left(i\omega + \frac{N^2}{i\omega}\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 p^{1/\Gamma_1}u_r\right) = -\frac{i\omega r^2 p'}{\Gamma_1 p^{1-1/\Gamma_1}}\left(1-\frac{S_l^2}{\omega^2}\right).$$

Here

$$S_l^2 = \frac{l(l+1)\Gamma_1 p}{\rho r^2} \quad \text{and} \quad N^2 = g\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln p}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right),$$

g being the gravitational acceleration. Derive, stating your assumptions, the equation governing p modes in the high ω limit in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2 p^{2/\Gamma_1}}{\rho}\frac{\mathrm{d}y}{\mathrm{d}r}\right) + \frac{r^2 y}{\Gamma_1 p^{1-2/\Gamma_1}}\left(\omega^2 - S_l^2\right) = 0, \qquad (*)$$

where

$$y = r^2 p^{1/\Gamma_1}u_r.$$

Derive a similar equation governing g modes in the low ω limit.

If y is everywhere regular and $p^{2/\Gamma_1}/\rho \to 0$ as $r \to R$, prove that the p mode spectrum derived from equation $(*)$ is stable.

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5 A fluid has motion and variations only in one spatial direction x. By appropriately combining the momentum and continuity equations in their standard form (with no external forces), and the adiabatic energy equation which you may assume in the form

$$\rho\frac{DU}{Dt} = \frac{p}{\rho}\frac{D\rho}{Dt},$$

U being the internal energy per unit mass, derive the momentum and energy equations in conservative form:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2+p) = 0,$$

$$\frac{\partial}{\partial t}\left(\rho U+\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x}\left(\rho u\left(U+\frac{p}{\rho}+\frac{1}{2}u^2\right)\right) = 0.$$

Hence deduce the jump conditions for a steady shock:

$$\begin{aligned}\rho_1 u_1 &= \rho_2 u_2\\ \rho_1 u_1^2+p_1 &= \rho_2 u_2^2+p_2\\ U_1+\frac{p_1}{\rho_1}+\frac{1}{2}u_1^2 &= U_2+\frac{p_2}{\rho_2}+\frac{1}{2}u_2^2\end{aligned}$$

where subscripts 1 and 2 denote conditions just upstream and just downstream of the shock.

For a perfect gas, $U=(\Gamma-1)^{-1}p/\rho$, where Γ is the adiabatic exponent. Show that for a strong shock, for which the upstream Mach number $M_1 \gg 1$, the jump conditions imply

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{\Gamma+1}{\Gamma-1} = \frac{u_1}{u_2}\\ \frac{p_2}{p_1} &= \frac{2\Gamma M_1^2}{\Gamma+1}.\end{aligned}$$

Consider the following highly simplified model based on the above. The Sun loses mass at a rate $\dot{M}$ in the form of the solar wind. At the orbit of the earth the wind is supersonic and the measured velocity is u_E. At a greater distance r_s from the Sun, the wind encounters a strong stationary shock. At an even greater distance, r_h, the wind encounters the heliopause, the boundary between the solar wind and the interstellar medium (which are assumed not to interpenetrate). Assume that the speed in the wind is constant in the supersonic regime. Deduce that the shock is located at

$$r_s = \left(\frac{\dot{M}}{\pi u_E \rho_2}\right)^{1/2}.$$

Between the shock and the heliopause, ρ and ρu^2+p are essentially constant, and the speed u declines rapidly with distance from the Sun, so that $\rho u^2+p=p_x$, where p_x is the pressure in the interstellar medium. Show that

$$r_s = \left(\frac{\dot{M}u_E}{4\pi p_x}\right)^{1/2}.$$

Assuming that $\dot{M}$ is constant over the Sun's main-sequence lifetime, show that the location of the heliopause varies slowly with time as

$$r_h \propto t^{1/3}.$$

Taking $\dot{M}=10^{-13}M_\odot$ per year, $u_E=4\times10^5\ \mathrm{m\,s^{-1}}$, the age of the Sun to be 5×10^9 years, and p_x to be $10^{-14}\ \mathrm{N\,m^{-2}}$ (appropriate to a temperature of 10^4K and a density of 10^5 atoms $\mathrm{m^{-3}}$), estimate the present positions of the shock and the heliopause.

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6 Write briefly on **three** of the following topics:

(a) effects of rotation in a solar-type star. Mixing in stellar interior;

(b) asteroseismology;

(c) spiral density waves in galactic disks;

(d) the effects of rotation and magnetic fields on star formation;

(e) "anomalous viscosity" in accretion disks;

(f) the dynamics of supernova remnants.

[End of paper]
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