Queen Mary College

UNIVERSITY OF LONDON

MAS 412 Relativity and Gravitation

21 May 10:00, 2001

Time Allowed: 3 Hours

Full marks may be obtained for complete answers to BOTH questions from section A plus ONE question from section B. There is no restriction on the number of questions that may be attempted but for grades A and B complete answers will be given substantially more credit than fragmentary ones. The use of calculators is permitted provided no use is made of programming, graph–plotting or algebraic facilities.

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
1 kpc		3.09×10^{19} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulas

The spherically symmetric Schwarzcshild metric interval is

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \frac{r_g}{r})} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right),$$

where $r_g = 2GM/c^2$ is the gravitational radius of the central body of the mass M.

The Riemann tensor is

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

Exam continued

SECTION A: Each question carries 35 marks. You should attempt both questions.

A1.

a) Explain the similarity between an "actual" gravitational field and a non-inertial reference system. Give the definition of a locally Galilean coordinate system. Explain why an "actual" gravitational field cannot be eliminated by any transformation of coordinates over all space-time.
Give the definition of a contravariant vector in terms of the transformation of curvilinear coordinates. Show with the help of straightforward differentiation that if A^i is a vector then dA^i is not a vector. [9 marks]

b) Show that in a uniformly rotating system of coordinates $x^{'}$, $y^{'}$, $z^{'}$, such that

$$x = x^{'} \cos \Omega t - y^{'} \sin \Omega t, \;\; y = x^{'} \sin \Omega t + y^{'} \cos \Omega t, \;\; z = z^{'},$$

the interval ds has the following form:

$$ds^2 = g_{ik} dx^i dx^k = g^{'}_{ik} dx^{'i} dx^{'k} =$$

$$[c^2 - \Omega^2 (x^{'2} + y^{'2})] dt^2 - dx^{'2} - dy^{'2} - dz^{'2} + 2\Omega y^{'} dx' dt - 2\Omega x^{'} dy' dt.$$

Give the definition of the contravariant metric tensor g^{ik}. Calculate the determinant $g^{'}$ of the matrix $g^{'}_{ik}$ to show that $g^{'} < 0$. [9 marks](unseen)

c) Motivate the necessity to introduce parallel translation of a vector. Explain the meaning of the Christoffel symbols,

$$\Gamma^i_{km} = \frac{1}{2} g^{in} \left(g_{kn,m} + g_{mn,k} - g_{km,n} \right),$$

and define the covariant derivative.
Explain what it means to say that the Christoffel symbols do not form a tensor. [9 marks]

d) Show by straightforward calculations that

$$\Gamma^i_{ki} = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \frac{\partial \ln \sqrt{-g}}{\partial x^k}.$$

(You can use here without proof that the differential of g can be expressed as $dg = g g^{ik} dg_{ik} = -g g_{ik} dg^{ik}$.) [3 marks]

e) Explain why for the derivation of physical equations in the presence of a gravitational field one can simply replace partial derivatives by covariant derivatives, and as an example show that the motion of a particle in a gravitational field is given by the geodesic equation

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0.$$

[5 marks]

Exam continued

A2.

a) Using a locally-inertial coordinate system prove that the Riemann tensor has the following symmetry properties:

i) $R_{iklm} = -R_{kilm} = -R_{ikml}$,

ii) $R_{iklm} = R_{lmik}$,

iii) $R_{iklm} + R_{imkl} + R_{ilmk} = 0$.

[9 marks]

b) Given that the Ricci tensor $R_{ik} = g^{lm} R_{limk}$ show that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

Using a locally-inertial coordinate system prove the Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Using this identity, prove that the Ricci tensor and the scalar curvature $R = g^{ik} R_{ik}$ satisfy the following identity:

$$R^l_{m;l} = \frac{1}{2}\frac{\partial R}{\partial x^m}.$$

[11 marks]

c) Using the Einstein equations in the form

$$R^i_k = \frac{8\pi G}{c^4}\left(T^i_k - \frac{1}{2}\delta^i_k T\right),$$

where G is the gravitational constant, and the identity obtained in c), prove that the energy-momentum tensor of matter T^i_k satisfies the conservation law:

$$T^k_{i;k} = 0.$$

[6 marks]

d) In the limiting case of a weak gravitational field described by a Newtonian potential ϕ we can write

$$g_{00} = 1 + \frac{2\phi}{c^2},$$

$g_{0\alpha} = 0$, $g_{\alpha\beta} = -\delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$.
Derive from the Einstein equations, using the $(0, 0)$ - component, that

$$\Delta\phi = 4\pi G\mu,$$

where μ is the density of matter. (Hint: in the nonrelativistic case $T^k_i = \mu c^2 u_i u^k$ and $u^\alpha = 0$, $u^0 = u_0 = 1$).

[9 marks]

Exam continued

SECTION B: Each question carries 30 marks. You should attempt one question.

B1.

a) A spherically symmetric gravitational field in vacuum is given by the Schwarzschild metric. Using this metric show that

$$\Gamma^0_{10} = -\Gamma^1_{11} = \frac{r_g}{2r^2}\left(1 - \frac{r_g}{r}\right)^{-1},$$

and

$$\Gamma^1_{00} = \frac{r_g}{2r^2}\left(1 - \frac{r_g}{r}\right).$$

(You may use the definition of Christoffel symbols given in A1 c) and the geodesic equation given in A1 e)).

[6 marks]

b) Show that the time component of the geodesic equation has the following form:

$$\frac{d^2t}{d\tau^2} + \frac{r_g}{r^2}\left(1 - \frac{r_g}{r}\right)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0,$$

where τ is proper time ($ds = cd\tau$). Solve this equation for a particle which is falling radially towards a black hole. Show that

$$\frac{dt}{d\tau}\left(1 - \frac{r_g}{r}\right) = 1.$$

[10 marks]

c) Show that when the particle in b) has zero velocity at infinity, then

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{c^2 r_g}{r}$$

.

[5 marks]

d) Find τ and t as functions of r and sketch them on the same graph. Explain why $t \to \infty$ when $r \to r_g$.
Consider a particle which has zero velocity at infinity and then falls towards a black hole. Find the proper time required to fall from $r = 10r_g$ to i) $r = 2r_g$, ii) $r = r_g$, and iii) $r = 0$. Express the results in seconds for a black hole with $M = 100M_\odot$, where $M_\odot$ is the solar mass.

[9 marks]

Exam continued

B2.

a) Explain briefly what is meant by a black hole and its event horizon. Describe how a distant observer sees the final stage of approach of some luminous body to the gravitational radius of the black hole.
A rotating black hole is described by the Kerr metric.

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$

$$\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the angular momentum of the black hole. Explain why the location of the event horizon, r_{hor}, is given by $g_{rr} = \infty$ and show that

$$r_{hor} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2}.$$

Compare this expression with the expression for the radius of the event horizon in the case of a non-rotating black hole.
[7 marks]

b) What is the limit of stationarity and what is the ergosphere? Explain why the location of the sphere corresponding to the limit of stationarity, r_{ls}, is determined by $g_{tt} = 0$, and show then that

$$r_{ls} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

Sketch the rotating black hole as projected on i) the equatorial plane, $\theta = \pi/2$, and ii) the perpendicular plane, $\phi = 0$ (indicate the event horizon, the limit of stationarity and the ergosphere).
[11 marks]

c) Explain qualitatively why it is possible to extract the energy of a rotating black hole despite the fact that no signal can escape outside from within the black hole horizon.
Show that the circle $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

(Hint: put $dr = 0$, $d\theta = 0$ and $d\phi = \Omega_{hor}dt$ into ds and show that $ds = 0$.)
[12 marks]

Exam continued

B3.

a) A plane gravitational wave propagates in the z direction. Consider a ring of particles initially at rest in the x-y plane. Sketch the distortions of the original ring for two states of polarization of the wave at $t = 0$, $t = T/4$, $t = T/2$, $t = 3T/4$ and $t = T$, where T is the period of the wave.

[5 marks]

b) Two objects of equal mass $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor, show that to order of magnitude

$$h \approx \frac{r_g}{R}\left(\frac{r\omega}{c}\right)^2,$$

where r is the distance between two objects, ω is the angular velocity of rotation of the objects around the centre of mass, $\omega = 2\pi/P$, r_g is the gravitational radius of the mass m and R is the distance to the binary system. Using the Newtonian relationship between r and ω,

$$Gm/r^2 = \omega^2 r/2,$$

show that

$$h = \frac{r_g}{R}\left(\frac{r_g\omega}{c}\right)^{2/3}.$$

[15 marks]

c) Future detectors of gravitational waves are supposed to have a sensitivity that will enable experimenters to detect gravitational waves with amplitudes as small as $h = 10^{-23}$. At what distance could such detectors detect gravitational waves emitted by a binary system with orbital period 8 min, where both masses are $20M_\odot$ ($M_\odot$ denotes solar mass)? To produce the estimate take into account that the gravitational radius of the Sun is equal to $3km$. Express the final answer in kiloparsecs ($1kpc = 10^3pc = 3 \cdot 10^{21}cm$).

[10 marks]

END OF EXAMINATION

A.Polnarev

Key objectives of "Relativity and Gravitation" MAS 412 (MTHM N64).

1. **Effects of general relativity in solar system and in the Universe:** you should have a good understanding of the importance of general relativity in physics and astronomy.

2. **Curvilinear coordinates, covariant differentiation:** You should be able to operate with concepts of differential geometry and understand the deep relationship between physics and geometry.

3. **Motion of particles in a gravitational field:** You should understand the fundamental difference in the motion of particles in relativistic theory of gravitation and in Newton's theory. You should be able to write down and solve in the simplest cases the geodesic equation.

4. **The curvature tensor and the Einstein equations:** You should understand basic physical principle of the least action and have good qualitative understanding of the most important stages of the derivation of these equations.

5. **Black holes:** You should understand what is event horizon and what is the limit of stationarity. You should be able to describe the main effects of strong gravitational field around black hole and have idea how the black holes could be discovered.

6. **Gravitational waves:** You should be able to derive the wave equation for propagation of gravitational radiation, understand why gravitational waves are transverse and traceless, what is similarity and what is the difference with electromagnetic waves. You should also be able to produce order of magnitude estimations of amplitudes of gravitational waves from astrophysical sources of gravitational radiation.

Syllabus of "Relativity and Gravitation" MAS 412 (MTHM N64).

1. Introduction to General Relativity.

2. Derivation from the basic principles of the Schwarzschild solution of the Einstein's field equations.

3. The Reisner-Nordstrom, Kerr and Kerr-Neuman solutions and physical aspects of the strong gravitational fields around black holes.

4. Generation, propagation and detection of gravitational waves.

5. Weak general relativistic effects in the Solar System and binary pulsars. Alternative theories of Gravity and experimental tests of General Relativity.