Queen Mary & Westfield College

UNIVERSITY OF LONDON

MAS 412 (MTHM N64) Relativity and Gravitation

24 May 10:00, 2000

Time Allowed: 3 Hours

Full marks may be obtained for complete answers to BOTH questions from section A plus ONE question from section B. There is no restriction on the number of questions that may be attempted but for grades A and B complete answers will be given substantially more credit than fragmentary ones. The use of calculators is permitted provided no use is made of programming, graph–plotting or algebraic facilities.

Physical Constants

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
1 kpc		3.09×10^{19} m

Notation

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature (+ - - -) is used.

Useful formulas

Except where specifically stated, the following results may be quoted without proof:

In an inertial reference system in Cartesian coordinates the metric interval is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

(Minkowski space-time).

Exam continued

The metric interval corresponding to a weak gravitational field has the following form:

$$ds^2 = (c^2 + 2\phi)dt^2 - dx^2 - dy^2 - dz^2,$$

where ϕ is the Newtonian potential.
The spherically symmetric Schwarzshild metric interval is

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \frac{r_g}{r})} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right),$$

where $r_g = 2GM/c^2$ is the gravitational radius of the central body of the mass M.
The Riemann tensor is

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

The Ricci tensor is

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

SECTION A: Each question carries 30 marks. You should attempt both questions.

A1.

a) Show that in a uniformly accelerated frame moving in Minkowski space-time along the x-axis with the system of coordinates $ct^{'}, x^{'}, y^{'}, z^{'}$ such that

$$x = x^{'} + \frac{at^{'2}}{2}, \;\; y = y^{'}, \;\; z = z^{'}, \;\; t = t^{'},$$

the metric interval ds has the following form:

$$ds^2 = g^{'}_{ik} dx^{'i} dx^{'k} = (c^2 - a^2 t^{'2}) dt^{'2} - 2at^{'} dx^{'} dt^{'} - dx^{'2} - dy^{'2} - dz^{'2}.$$

Calculate the determinant $g^{'}$ of the matrix $g^{'}_{ik}$ to show that $g^{'} < 0$.
b) Give the definition of the contravariant metric tensor g^{ik}. Show that $g^{'00} = 1$, $g^{'11} = -1 + a^2 t^{'2}/c^2$, and $g^{'01} = -at^{'}/c$.
c) Suppose a light signal propagates from a point B with coordinates $(x^{'} + \Delta x^{'}, 0, 0)$ to a point A with coordinate $(x^{'}, 0, 0)$ and then back to the point B over the same path. If $t^{'}_0$ is the moment of arrival of the signal at A, find the time when it left B and when it returned to B. Calculate the corresponding interval of proper time $\Delta\tau$ for an observer at B between the moments when the signal left B and returned to B. Calculate the physical distance Δl between the points A and B.

Question continued

d) Consider the same points A and B as in section (c). Some event takes place at time $t_B^{'}$ at B. Show that time $t_A^{'}$ of the simultaneous event at A is given by

$$t_A^{'} = t_B^{'} + \frac{at\Delta x^{'}}{c(1 - a^2 t_0^{'2}/c^2)},$$

where $t_0^{'}$ is defined in section (c). Discuss briefly the problem of synchronization of clocks located at different points of space-time.

e) Repeat all calculations of section (c) for the case where the points A and B are located not in the non-inertial frame of section (a), but in a weak gravitational field with the Newtonian potential ϕ.

A2.

a) The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that $u_i u^i = 1$ and $p_i p^i = m^2c^2$. Show that in a static gravitational field with metric interval $ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, the energy of the particle, $E = mc^2 u_0$, is given by

$$E = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where

$$v = \frac{c\sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}}{\sqrt{g_{00}}dx^0}.$$

b) Consider the motion of a particle in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzshild gravitational field. Discuss why in this gravitational field the energy E and angular momentum $L = mcu_3$ of the particle are constants. Show that

$$\left(1 - \frac{r_g}{r}\right)^{-1}\left(\frac{E}{mc^2}\right)^2 - \frac{1}{r^2}\left(\frac{L}{mc}\right)^2 - \left(1 - \frac{r_g}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 = 1.$$

The "the effective potential energy" is defined by

$$U(r) = mc^2\left(1 - \frac{r_g}{r}\right)^{1/2}\left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2}.$$

Explain why the condition $E > U(r)$ determines the admissible range of the motion. Sketch $U(r)$ as a function of r for each of the values of the angular momentum $L = 0,\ \sqrt{3}mcr_g,\ 2mcr_g,\ 6mcr_g$.

c) Give qualitative arguments to show that the minima of $U(r)$ correspond to stable circular orbits and the maxima correspond to unstable circular orbits. Solve the simultaneous equations $U(r) = E$ and $U^{'}(r) = 0$ to find the radius $r_c(L)$ of the stable circular orbit as a function of L. Show that E, L and r_c are related by

$$E = Lc\sqrt{\frac{2}{r_c r_g}}\left(1 - \frac{r_g}{r_c}\right).$$

Question continued

d) Show that the stable circular orbit closest to the origin has the parameters $r_c = 3r_g$, $L = \sqrt{3}mcr_g$, $E = (2\sqrt{2}/3)mc^2$. A particle with $E_0 = mc^2$ at infinity moves along a sequence of circular orbits with slowly decreasing radius. What fraction of the initial energy will be released by the particle when it reaches the closest circular orbit? (Compare this with the case of an H-bomb, given that this fraction in that case is about 0.8%.)

SECTION B: Each question carries 40 marks. You should attempt one question.

B1.

a) To make clear the true character of the Schwarzschild space-time metric at $r = r_g$, make a transformation of the coordinates of the following form:

$$c\tau = ct + \int \frac{f(r)dr}{1 - \frac{r_g}{r}}, \quad R = ct + \int \frac{dr}{\left(1 - \frac{r_g}{r}\right) f(r)},$$

where $f(r)$ is an arbitrary differentiable function. Show that after this transformation the Schwarzschild metric takes the form:

$$ds^2 = \frac{1 - \frac{r_g}{r}}{1 - f^2}(c^2 d\tau^2 - f^2 dR^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Choose the function $f(r)$ so as to eliminate the singularity at $r = r_g$ and make $g_{00} = 1$ there. Hence show that

$$r = r_g^{1/3}\left[\frac{3}{2}(R - c\tau)\right]^{2/3}.$$

By substituting this expression for r into the above metric, demonstrate that the final metric is non-stationary.

b) Consider the propagation of radial light signals in the above metric. Show that the equation $ds = 0$ for $\theta,\ \phi = const$ gives

$$\frac{d(c\tau)}{dR} = \pm\sqrt{\frac{r_g}{r}}.$$

Using the $(c\tau, R)$ diagram, show that the line $r = const$ falls inside the light cone for $r > r_g$, but outside it for $r < r_g$. Discuss very briefly how these results can be applied to the problem of gravitational collapse in the general theory of relativity.

c) Give the definition of the event horizon. Using the Schwarzschild metric, calculate the time Δt for the propagation of signals from the surface of a collapsing body at r to some point with $r_0 > r$. Show that Δt diverges as $r \to r_g$.

Exam continued

B2.

a) Explain how one obtains the differential of a vector which itself behaves like a vector? Show that the covariant derivatives are given by

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k$$

and

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k.$$

Show that the Christoffel symbols Γ^i_{kl} are symmetric with respect to the lower indices: $\Gamma^i_{kl} = \Gamma^i_{lk}$. Prove that $g_{ik;l} = 0$ and hence show that

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

Infer that

$$\Gamma^i_{ki} = \frac{\partial \ln \sqrt{-g}}{\partial x^k}.$$

(Hint: you may assume without proof that $dg = g g^{ik} dg_{ik} = -g g_{ik} dg^{ik}$.)

b) Show that

$$A_{i;k;l} - A_{i;l;k} = A_m R^m_{ikl},$$

where R^i_{klm} is the Riemann tensor.

c) Consider two neighbouring particles freely falling from rest in the Schwarzschild gravitational field. Using the geodesic deviation equation

$$\frac{D^2 \eta^i}{c^2 d\tau^2} = R^i_{klm} u^k u^l \eta^m,$$

where τ is the proper time, u^k is the four-velocity and η^i is the separation vector between the particles, show that the component of the Riemann tensor which is responsible for the tidal force in the radial direction is

$$R^1_{001} = \frac{r_g}{r^3} \left(1 - \frac{r_g}{r} \right).$$

If the mass and height of an observer are 80kg and 2m respectively, find the radial distance r from a solar mass neutron star at which the tidal force experienced by the observer at rest exceeds 1600N. You may assume that the observer's body is aligned along radial direction and you may take the gravitational radius of the Sun to be 3 km.

Exam continued

B3.

a)Consider a weak vacuum gravitational field. The tensor h_{ik} describes a small perturbation of the Galilean metric $g_{ik}^{(0)}$: $g_{ik} = g_{ik}^{(0)} + h_{ik}$. Show that $g^{ik} = g^{(0)ik} - h^{ik}$where $h^{ik} = g^{(0)im} g^{(0)kn} h_{mn}$. The condition that h_{ik} be small leaves the possibility of an arbitrary transformation of coordinates of the form $x^{'i} = x^i + \xi^i$, with small ξ^i. Show that after such a transformation

$$h_{ik}^{'} = h_{ik} - \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i}.$$

By imposing the supplementary condition $h_{i;l}^{k} - \frac{1}{2}\delta_i^k h_k = 0$, where $h = h_i^i$, calculate the Ricci tensor, considering only linear terms with respect to h_{ik}. Show that the perturbation tensor h_{ik} satisfies the wave equation:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})h_{ik} = 0,$$

where ∇^2 is the 3-dimensional Laplacian operator.

b) Consider a plane gravitational wave propagating along the x-axis. Show that if all the quantities h_k^i are functions of $(t - x/c)$, then they satisfy the wave equation. By using the additional coordinate transformation $x^{'i} = x^i + q^i(t - x/c)$, together with the previous supplementary condition, infer that one can make all components of h_k^i vanish except $h_{22} = -h_{33} \equiv h_+$ and $h_{23} = h_{32} = h_\times$.

Consider two test particles located in the (y, z) plane, separated by the 3-vector $l^\alpha = (0, l_0 \cos\theta, l_0 \sin\theta)$. Show that the perturbation of the distance δl between the two particles in the gravitational wave varies as

$$\delta l = l - l_0 = \frac{l_0}{2}(h_+ \cos 2\theta + h_\times \sin 2\theta).$$

c) Consider a ring of test particles initially at rest in the (y, z)-plane and a plane monochromatic gravitational wave with frequency ω and polarization $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$. Sketch the shape of the ring perturbed by the gravitational wave at times $t = \frac{\pi}{2\omega}$, $\frac{\pi}{\omega}$, $\frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$.

Repeat the analysis for a gravitational wave with another polarization: $h_+ = 0$, $h_\times \sin\omega(t - x/c)$. Compare the result with the previous one. Finally consider the following superposition of two polarized waves: $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = h_0 \cos\omega(t - x/c)$. Sketch the shape of the perturbed ring in this case and compare your result with the two previous ones. What would you call this state of polarization?

d) The quadrupole formula for the generation of gravitational waves is given by

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where G is the gravitational constant, R is the distance to the source of the gravitational waves and $D_{\alpha\beta} = \int(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$ is the quadrupole tensor. Consider a binary system consisting of two components of the same mass m. Show that to an order of magnitude

$$\frac{d^2 D_{\alpha\beta}}{dt^2} \sim mr^2(2\omega)^2\cos(2\omega + \phi),$$

Question continued

where ω is the orbital angular velocity, r is the separation distance between the two components and ϕ is an arbitrary phase. Estimating ω from Newton' laws, show that to an order of magnitude the amplitude of the gravitational waves is $h_0 \sim r_g^2/(rR)$, where r_g is the gravitational radius of each component.

e) Imagine some future gravitational wave antenna able to detect gravitational waves with frequency $10^{-3} Hz$ and amplitude exeeding 10^{-23}. Estimate the mass m and the separation r between the two components of an equal mass binary system located in the centre of our Galaxy ($R \approx 10kpc$), in order that the gravitational waves be detectable.

END OF EXAMINATION

A.Polnarev

Key objectives of "Relativity and Gravitation" MAS 412 (MTHM N64).

1. **Effects of general relativity in solar system and in the Universe:** you should have a good understanding of the importance of general relativity in physics and astronomy.

2. **Curvilinear coordinates, covariant differentiation:** You should be able to operate with concepts of differential geometry and understand the deep relationship between physics and geometry.

3. **Motion of particles in a gravitational field:** You should understand the fundamental difference in the motion of particles in relativistic theory of gravitation and in Newton's theory. You should be able to write down and solve in the simplest cases the geodesic equation.

4. **The curvature tensor and the Einstein equations:** You should understand basic physical principle of the least action and have good qualitative understanding of the most important stages of the derivation of these equations.

5. **Black holes:** You should understand what is event horizon and what is the limit of stationarity. You should be able to describe the main effects of strong gravitational field around black hole and have idea how the black holes could be discovered.

6. **Gravitational waves:** You should be able to derive the wave equation for propagation of gravitational radiation, understand why gravitational waves are transverse and traceless, what is similarity and what is the difference with electromagnetic waves. You should also be able to produce order of magnitude estimations of amplitudes of gravitational waves from astrophysical sources of gravitational radiation.

Syllabus of "Relativity and Gravitation" MAS 412 (MTHM N64).

1. Introduction to General Relativity.

2. Derivation from the basic principles of the Schwarzschild solution of the Einstein's field equations.

3. The Reisner-Nordstrom, Kerr and Kerr-Neuman solutions and physical aspects of the strong gravitational fields around black holes.

4. Generation, propagation and detection of gravitational waves.

5. Weak general relativistic effects in the Solar System and binary pulsars. Alternative theories of Gravity and experimental tests of General Relativity.