Queen Mary & Westfield College

UNIVERSITY OF LONDON

MAS 412 (MTHM N64) Relativity and Gravitation

25 May 10:00, 1999

Time Allowed: 3 Hours

Full marks may be obtained for complete answers to TWO questions from section A plus ONE question from section B. There is no restriction on the number of questions that may be attempted but for grades A and B complete answers will be given substantially more credit than fragmentary ones. The use of calculators is permitted provided no use is made of programming, graph–plotting or algebraic facilities.

SECTION A: Each question carries 30 marks. You should attempt both questions.

A1.

a) Explain what is the similarity between an "actual" gravitational field and a non-inertial reference system and what is the difference.
b) Consider an observer moving along the x^1-axis in an arbitrary space-time $ds^2 = g_{ik}dx^i dx^k$. His world line is given in a parametric form

$$x^1 = f(\lambda), \;\; x^0 = ct = F(\lambda),$$

where λ is some arbitrary parameter along this world line. Show that the proper time of the observer (clock time on his wrist watch) is related to λ by the following relation:

$$\tau = \frac{1}{c}\int \sqrt{g_{00}(\frac{dF}{d\lambda})^2 + 2g_{01}\frac{dF}{d\lambda}\frac{df}{d\lambda} + g_{11}(\frac{df}{d\lambda})^2 d\lambda}$$

Compare this result with the case of an observer at rest. Show that in this case

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0$$

What is the world line of such observer in terms of the functions $F(\lambda)$ and $f(\lambda)$?
c) Give the definition of a covariant vector in terms of the transformation of curvilinear coordinates and compare this transformation with the transformation of the gradient of a scalar function Φ. Show with the help of straightforward differentiation that if $\frac{\partial \Phi}{\partial x^i}$ is a vector then $d\frac{\partial \Phi}{\partial x^i}$ is not a vector.

continued

d) Using the definition of covariant derivatives

$$A^i_{;l} = A^i_{,l} + \Gamma^i_{ml} A^m, \quad A_{i;l} = A_{i,l} - \Gamma^m_{il} A_m,$$

where Γ^i_{km} are the Christoffel symbols and "," means partial differentiation, show that $g_{ik;l} = 0$ and thus that

$$\Gamma^i_{km} = \frac{1}{2} g^{in} \left(g_{kn,m} + g_{mn,k} - g_{km,n} \right).$$

e) Show by straightforward calculation that the differential of g, where g is the determinant of the metric tensor g_{ik}, can be expressed as follows

$$dg = g g^{ik} dg_{ik} = -g g_{ik} dg^{ik}.$$

Then show that

$$\Gamma^i_{ki} = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \frac{\partial \ln \sqrt{-g}}{\partial x^k},$$

$$A^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} A^i)}{\partial x^i},$$

$$\Phi^{;i}_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} g^{ik} \frac{\partial \Phi}{\partial x^k} \right),$$

where Φ is a scalar.

f) Explain why in physical equations in the presence of a gravitational field one can simply replace partial derivatives by covariant derivatives. As an example show that the equation of conservation of electric charge in the presence of a gravitational field can be written in the form

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} j^i}{\partial x^i} = 0,$$

where j^i is the 4-vector of electric current.

A2.

a) Using the following for the Riemann curvature tensor

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{pm} \Gamma^p_{kl},$$

show that

$$R_{iklm} = -R_{kilm} = -R_{ikml} = R_{lmik}$$

b) Show that

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0$$

(Bianchi identity) (Hint: Use Galilean coordinates, where $\Gamma^i_{km} = 0$, but the $\frac{\partial \Gamma^i_{km}}{\partial x^l}$ are not all zero).

continued

c) In an inertial reference system in polar coordinates the interval ds is given by the relation:

$$ds^2 = c^2 dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Explain why a uniformly rotating system of coordinates, t,r, θ, ϕ' such that

$$\phi = \phi' + \Omega t,$$

is non-inertial. Show that the interval ds has the form

$$ds^2 = g_{ik}dx^i dx^k = (c^2 - \Omega^2 r^2 \sin^2\theta)dt^2 - 2\Omega r^2 \sin^2\theta dt d\phi' - dr^2 - r^2(d\theta^2 - \sin^2\theta d\phi'^2).$$

d) Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

SECTION B: Each question carries 40 marks. You should attempt one question.

B1.

a) A spherically symmetric gravitational field in vacuum is given by the Schwarzshild metric,

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + sin^2\theta d\phi^2),$$

where $r_g = \frac{2km}{c^2}$ is the gravitational radius. Using this metric show that:

$$\Gamma^0_{10} = -\Gamma^1_{11} = \frac{r_g}{2r^2}\left(1 - \frac{r_g}{r}\right)^{-1}$$

$$\Gamma^1_{00} = \frac{r_g}{2r^2}\left(1 - \frac{r_g}{r}\right)$$

$$\Gamma^1_{22} = \Gamma^1_{33}/\sin^2\theta = -r\left(1 - \frac{r_g}{r}\right)$$

$$\Gamma^2_{21} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\frac{1}{2}\sin 2\theta$$

$$\Gamma^3_{23} = \cot\theta.$$

Show that all the other symbols vanish or can be obtained from these by symmetry.

continued

b) Show that the time-component of the geodesic equation has the following form:

$$\frac{d^2t}{d\tau^2} + \frac{r_g}{r^2}\left(1 - \frac{r_g}{r}\right)^{-1} \frac{dt}{d\tau}\frac{dr}{d\tau} = 0,$$

where τ is proper time ($ds = cd\tau$). Assume that at infinity $d\tau = dt$.What does this imply about the velocity at infinity? From this equation show that

$$\frac{dt}{d\tau}\left(1 - \frac{r_g}{r}\right) = 1.$$

Using the expression for ds show that in this case

$$\frac{dr}{d\tau} = -c\left(\frac{r_g}{r}\right)^{1/2}$$

A particle falls radially towards the horizon of a Scharzschild black hole. Find τ as a function of r. Taking $t = \tau = 0$, when $r = 3r_g$, show that $t \to \infty$ when $r \to r_g$. Find the proper time required to reach $r = 2r_g$, $r = r_g$ and $r = 0$. Express your results in seconds for $M = 10M_\odot$, where $M_\odot$ is the solar mass.

B2.

a) A rotating black hole is described by the Kerr metric

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$
$$\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the angular momentum of the black hole.

Show that if $a = 0$ (the non-rotating black hole) the Kerr metric is reduced to the Schwarzshild metric.

Explain why the location of the event horizon, r_{hor}, is given by $g_{rr} = \infty$ and show that

$$r_{hor} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2}.$$

Compare this expression with that for the event horizon in the case of a non-rotating black hole.

b) Show that the circle $r = r_{hor}$, $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

(Hint: put $dr = 0$, $d\theta = 0$ and $d\phi = \Omega_{hor}dt$ into the equation for ds with $ds = 0$.)

continued

c) What is the limit of stationarity and what is the "ergosphere"? Explain why the location of the sphere corresponding to the limit of stationarity, r_{ls}, is determined by $g_{tt} = 0$, and show then that

$$r_{ls} = \frac{r_g}{2} + \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2 \theta}.$$

Explain qualitatively why and how it is possible to extract the energy of a rotating black hole despite the fact that no signal can escape outside from the black hole horizon.

d) Sketch the rotating black hole in two projections: i) in the equatorial plane, $\theta = \pi/2$, ii) in the perpendicular plane, $\phi = 0$.

B3.

a) Describe the procedure to determine the spatial distance dl between two infinitesimally close observers located at two points $A(x^\alpha)$ and $B(x^\alpha + dx^\alpha)$.
Sketch the world lines of the signal used for the determination of distance. Show that

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}\right) dx^\alpha dx^\beta (\alpha, \beta = 1, 2, 3).$$

b) Consider the case when the gravitational field is weak, for example far from the gravitating body. Take the almost galilean metric tensor in the form

$$g_{ik} = g_{ik}^{(0)} + h_{ik},$$

where

$$g_{00}^{(0)} = 1, \;\; g_{0\alpha}^{(0)} = 0, \;\; g_{\alpha\beta}^{(0)} = -\delta_{\alpha\beta}$$

and the h_{ik} are small corrections determined by the gravitational field. Using the Einstein equations in the form

$$R_k^i = \frac{8\pi\kappa}{c^4}(T_k^i - \frac{1}{2} g_k^i T),$$

where κ is the gravitational constant, show that in vacuum

$$-g^{lm(0)} \frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \frac{\partial^2 h_i^l}{\partial x^k \partial x^l} + \frac{\partial^2 h_k^l}{\partial x^i \partial x^l} - \frac{\partial^2 h}{\partial x^i \partial x^k} = 0,$$

where $h_i^k = g^{(0)kl} h_{il}$ and $h = h_i^i$.
Simplify this equation by making use of the freedom in the choice of reference frame. Impose on the h_{ik} four supplementary conditions

$$\frac{\partial (h_k^i - \frac{1}{2}\delta_k^i h)}{\partial x^i} = 0$$

and show that

$$g^{lm(0)} \frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} = 0.$$

Show explicitly that this is the wave equation.

continued

c) The metric tensor corresponding to a weak gravitational wave has the following form:

$$g_{ik} = \eta_{ik} + h_{ik},$$

where $\eta_{00} = 1$, $\eta_{\alpha\beta} = -\delta_{\alpha\beta}$, $h_{23} = h_{32} = h_\times$ and $h_{22} = -h_{33} = h_+$.
Show that the distance δl between two test particles in the gravitational wave is given by the following expression

$$\delta l = \delta l_0 (1 - \frac{1}{2}(h_+[(n^2)^2 - (n^3)^2] + 2h_\times n^2 n^3),$$

where δ_0 is the unperturbed distance between two test particles and n^α is the unit 3-vector such that $\delta x^\alpha = n^\alpha \delta l_0$.

d) A black hole of mass $m = 100 M_\odot$ moves in a circular orbit around a super-massive black hole of mass $M = 10^6 M_\odot$ situated at the centre of our Galaxy. The radius of the orbit is $r = 100 r_g$ (r_g is the gravitational radius of the super-massive black hole). The quadrupole formula for the generation of gravitational waves gives

$$h_{\alpha\beta} = -\frac{2\kappa}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where κ is the gravitational constant and

$$D_{\alpha\beta} = m(3x_\alpha x_\beta - r^2 \delta_{\alpha\beta})$$

is the quadrupole tensor. Choose Cartesian coordinates with origin located in the centre of the super-massive black hole, and let $x^1 = R$, $x^2 = x^3 = 0$ correspond to the position of the Solar system. Assume for simplicity that the orbit lies in the (x^2, x^3) -plane. Show that

$$D_{22} = \frac{1}{2} m r^2 (1 + 3\cos 2\omega t),$$

$$D_{33} = \frac{1}{2} m r^2 (1 - 3\cos 2\omega t),$$

$$D_{23} = \frac{3}{2} m r^2 \sin 2\omega t,$$

where ω is the orbital angular velocity. Evaluate ω using Newton's laws in terms of M and r.
Show that the components of the gravitational wave are equal to

$$h_{22} = -h_{33} = \frac{r_g^2}{rR} \cdot \frac{m}{M} \cdot \cos 2\omega(t - R/c),$$

$$h_{23} = -\frac{r_g^2}{rR} \cdot \frac{m}{M} \cdot \sin 2\omega(t - R/c).$$

e) Consider two spacecraft in the Solar system separated by the distance $\delta l_0 = 5 \cdot 10^6 km$. Estimate the amplitude of perturbations in the separation between the satellites produced by the gravitational wave from the the system of two black holes mentioned above (take $R = 10 kpc$ and $n^\alpha = (0, 1, 0)$).

END OF EXAMINATION

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