Queen Mary & Westfield College
UNIVERSITY OF LONDON

ASTM108/MTHMN61 Cosmology
MAS 401 Advanced Cosmology

June 2nd 1999 10 am

Time allowed: 3 hours

Full marks may be obtained by answering about THREE complete questions. More credit will be given for attempting complete questions than fragments of questions. All questions carry equal marks. The use of an electronic calculator is permitted providing you do not make use of any algebraic, programming or graph-plotting facilities.

The following constants may be assumed:

Speed of light	$c=3.0\times10^{8}$ m s^{-1}
Gravitational constant	$G=6.7\times10^{-11}$ N m^{2}kg^{-2}
Mass of proton	$m_p=1.7\times10^{-27}$ kg
Planck's constant	$\hbar=1.1\times10^{-34}$ J s
Boltzmann constant	$k=1.4\times10^{-23}$ J K^{-1}
Black-body constant	$a=7.6\times10^{-16}$ J m^{-3}K^{-4}
Hubble time	$H_0^{-1}=9.8\times10^{9}h^{-1}$y
Electron volt	$1\text{eV}=1.6\times10^{-19}$ J

The following formulae may be assumed:

Robertson-Walker metric

$$ds^2 = c^2dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \qquad (k=+1,0,-1)$$

Evolution equations for scale factor

$$\dot{a}^2 + kc^2 = (8\pi/3)G\rho a^2 + \Lambda a^2c^2/3$$

$$\ddot{a} = -(4\pi/3)G(\rho + 3p/c^2)\, a + \Lambda ac^2/3$$

Distribution function for species i

$$f_i(p,t) = \frac{g_i}{(2\pi\hbar)^3}\exp\left[\frac{E-\mu_i}{kT_i} \pm 1\right]^{-1}, \quad E = \sqrt{p^2c^2 + m^2c^4} \quad \left(\begin{matrix}\text{fermions}\\ \text{bosons}\end{matrix}\right)$$

Question 1

(a) Show that the Robertson-Walker metric (given in the rubric in terms of the radial coordinate r and the curvature constant k) can be written as

$$ds^2 = c^2dt^2 - a(t)^2 [d\chi^2 + (A^{-1}\sin A\chi)^2 (d\theta^2 + \sin^2\theta \, d\phi^2)]$$

where χ is a new radial coordinate and the constant A has three possible values. Express χ in terms of r and also indicate the relationship between k and A. Give the form of the metric explicitly for the three values of A.

(b) Calculate the proper area of the sphere with constant radial coordinate at time t, expressing this in terms of both r and χ. Show that this is less than the Euclidean value for k=+1 and more than it for k=-1.

(c) By drawing the past light-cone for an observer at the present epoch, explain why a galaxy with comoving coordinate χ is observed at an earlier time t and indicate how one uses the Robertson-Walker metric to calculate t in terms of χ. For a pressureless model with k=0, show that the physical distance of a galaxy at the time t when it is observed can be expressed as

$$l(t) = 3c(t^{2/3}t_0^{1/3} - t) \ .$$

Infer the form of the past light-cone from t=0 until the present epoch and illustrate this with a diagram. (You may assume $a \propto t^{2/3}$ in this case.)

(d) By considering successive photons emitted from a galaxy at time t and received at the present time t_0, show that the light is redshifted by an amount

$$z = a_0/a(t) - 1$$

where a_0 is the current value of a(t).

(e) Show that the apparent angular size of a galaxy with linear diameter D, comoving radial coordinate χ and redshift z is

$$\theta = D(1+z)/[a_0A^{-1}\sin A\chi] \ .$$

In a Friedmann model with $\Lambda=0$ and p=0, you are given that

$$a_0A^{-1}\sin A\chi = 2cH_0^{-1}\Omega_0^{-2}(1+z)^{-1}[\Omega_0 z + (\Omega_0-2)(\sqrt{\Omega_0 z + 1} - 1)] \ ,$$

where Ω_0 is the current density parameter. If $\Omega_0=1$, deduce that θ has a minimum and calculate the redshift at which this occurs. If $\Omega_0 \ll 1$, prove that θ decreases as z^{-1} for $z \ll 1$, remains roughly constant for $1 \ll z \ll \Omega_0^{-1}$ and then increases as z for $z \gg \Omega_0^{-1}$. Discuss briefly how this can be used as an observational probe of the density parameter.

Question 2

(a) Explain briefly why the contents of the Universe can be regarded as a fluid at both late and early times. In what circumstances can it be regarded as a perfect fluid and when is this assumption likely to fail? For a Friedmann cosmological model, what is meant by the density parameter Ω and the deceleration parameter q?

(b) If the Universe consists of a perfect fluid with equation of state $p=\alpha\rho c^2$, show that ρ scales as $a^{-3(1+\alpha)}$. Infer that the Friedmann equation (given in the rubric) with $\Lambda=0$ can be written in the form

$$\dot{a}^2 = a_o^2 H_o^2[\Omega_o(a_o/a)^{1+3\alpha} + 1 - \Omega_o]$$

where a subscript o indicates values at the present epoch. Deduce that the density parameter evolves as

$$(1/\Omega) - 1 = [(1/\Omega_o)-1](1+z)^{-1-3\alpha} \ .$$

Use the equation for $\ddot{a}$ (given in the rubric) for this model to derive the values of α for which there is an initial Big Bang singularity. Infer that any perfect fluid model which starts with a Big Bang necessarily has $\Omega \rightarrow 1$ at early times.

(c) In a k=-1 Friedmann model prove that the k term in the equation for a is negligible for redshifts exceeding some value z_* and calculate this redshift in terms of Ω_o and α. For $z \gg z_*$, show that the dependence of a upon t and the density are given approximately by

$$a \propto t^{2/[3(1+\alpha)]} \quad , \quad \rho = 1/[6(1+\alpha)^2\pi G t^2] \ .$$

Also derive an expression for the deceleration parameter in this period.

(d) In Milne's special relativistic kinematic model, galaxies move with different fixed speeds u relative to some observer O. Their world-lines are therefore the straight lines $\{x=u\tau\}$ and confined to the wedge $|x|<c\tau$, where x and τ are special relativistic space and time coordinates. In terms of the proper time t measured by each galaxy, show that the world-lines have the form

$$x = c\,(\tau^2 - t^2)^{1/2} \ .$$

Infer that the hypersurfaces of constant t are hyperbolae. Explain why the late stages of a k=-1 Friedmann model (in which the spatial hypersurfaces have negative curvature) can be described by the Milne model (in which the spacetime is flat).

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Question 3

(a) What is meant by the distribution function for a particle species? In thermal equilibrium this is given by the expression in the rubric. Explain the significance of the factors g_i and μ_i in this expression. Show that this leads to the following expressions for the number density and mass density of relativistic species:

$$n \approx \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} \frac{2}{\pi^2} \zeta(3) \frac{g}{2} \left(\frac{kT}{\hbar c} \right)^3, \quad \rho \approx \begin{pmatrix} 7/8 \\ 1 \end{pmatrix} \frac{g}{2} aT^4 \quad \begin{pmatrix} \text{fermions} \\ \text{bosons} \end{pmatrix}$$

where $a = \pi^2 k^4/(15\hbar^3 c^3)$ is the black-body constant and $\zeta(3)$ is the Riemann zeta function. (There is no need to *derive* the numerical constants but you should indicate how they arise and why they are different for bosons and fermions.) Indicate the values of g for photons, electrons and neutrinos.

(b) Deduce that the early Universe can be regarded as comprising many species of black-body radiation, with the effective number of species being expressed as

$$N(T) = \sum_B (g_B/2)(T_B/T)^4 + (7/8) \sum_F (g_F/2)(T_F/T)^4$$

where T is the photon temperature and the sum is over boson species (B) and fermion species (F). Indicate qualitatively why T_B and T_F can differ from T.

(c) Use the Friedmann equation (given in the rubric) with Λ=O to show that the temperature in the early Universe evolves according to

$$T \approx N^{-1/4}(t/s)^{-1/2} \text{ MeV}.$$

[You may assume the results given in Question 2(c) with the appropriate value of α.] Infer the approximate times at which electrons and muons are pair-produced, assuming that their rest masses are 0.5 MeV and 106 MeV respectively. If the number of neutrino species is N_ν, determine exact expressions for N(T) during the radiation and lepton eras.

(d) Given that the entropy density of a component of the Universe with density ρ and pressure p is $s=(\rho c^2+p)T^{-1}$ and that the total entropy density is conserved during the annihilation of electrons and positrons, show that the temperature of any surviving neutrinos is $(4/11)^{1/3}$ times the temperature of the microwave background photons. Infer that their present number density is 3/11 times the number density of the photons.

(e) If electron-neutrinos have non-zero rest mass, show they can have the critical density providing this mass is around $100h^2$eV, where h is the Hubble parameter in unit of 100 km $s^{-1}Mpc^{-1}$. [You may assume that the neutrinos are non-relativistic today but relativistic when they decoupled; you will need to calculate the critical density in terms of h.]

Question 4

(a) Explain what is meant by the Hubble radius and the particle horizon size and explain their physical significance. Calculate both of these for a Friedmann model with equation of state $p=\alpha\rho c^2$, indicating the range of values of α for which your expressions are valid. [You may assume the form of the Robertson-Walker metric in the rubric and the evolution law for the scale factor given in Question 2(c).]

(b) Show that the comoving scale associated with the Hubble radius at time t scales as

$$r_c \propto a(t)^{(1+3\alpha)/2} .$$

Explain how this motivates the inflationary scenario and indicate the range of value of α for which this occurs. If the Universe undergoes an inflationary phase between times t_i and t_f, deduce that the current particle horizon was causally connected by the end of inflation providing

$$(a_f/a_i)^{(1+3\alpha)} >> 10^{63} z_{eq}^{-1}(T_f/T_{Pl})^2$$

where T_f is the temperature at the end of inflation (after reheating), T_{Pl} is the Planck temperature (whose value should be indicated), and z_{eq} is the redshift of matter-radiation equality. Deduce a lower limit for the number of expansion e-folds during inflation in terms of α and T_f.

(c) Discuss briefly what is meant by Grand Unification. Indicate roughly the mass-scale M_X associated with this unification, the time after the Big Bang of the related phase transition and the value of the unification coupling constant α_X. [You may assume the relationship between temperature and time given in Question 3(c).] Explain the physical significance of the Planck mass M_{Pl} and indicate its approximate value.

(d) If the Grand Unification phase transition produces one magnetic monopole per horizon volume, each with mass $\alpha_X^{-1}M_X$, show that the monopole-to-photon ratio is roughly $(M_X/M_{Pl})^3$. Deduce that the present monopole density would be incompatible with observation in the simplest Friedmann model. How can the inflationary scenario resolve this problem?

Question 5

(a) In a Newtonian cosmological model, the equations for an adiabatic self-gravitating adiabatic fluid are

$$\partial\rho/\partial t + \nabla.(\rho \mathbf{v}) = 0$$

$$\partial \mathbf{v}/\partial t + (\mathbf{v}.\nabla)\mathbf{v} + (\nabla p)/\rho + \nabla\phi = 0$$

$$\nabla^2\phi = 4\pi G\rho$$

where ρ is the density, $\mathbf{v}$ is the velocity and ϕ is the gravitational potential. Explain briefly the physical significance of each of these equations.

(b) The unperturbed background is assumed to be an expanding homogeneous model with scale factor a(t), density $\rho=\rho_1(t)$, pressure $p=p_1(t)$ and gravitational potential $\phi=\phi_1(t)$. Show that the first equation implies $\rho_1 \propto a^{-3}$ and infer the form of ϕ_1 from the last equation.

(c) Consider an adiabatic perturbation to this background of the form

$$\delta(\rho, \mathbf{v}, \phi) = (D, \mathbf{V}, \Phi) \exp(i\mathbf{k}.\mathbf{r}),$$

where D, **V** and Φ depend only on time. Show that to first order in the perturbed quantities, the following equations apply:

$$\dot{D} + 3(\dot{a}/a)D + i\rho\mathbf{k}.\mathbf{V} = 0$$

$$\dot{\mathbf{V}} + (\dot{a}/a)\,\mathbf{V} + iv_s^2 D\mathbf{k}/\rho + i\mathbf{k}\Phi = 0$$

$$k^2\Phi + 4\pi G D = 0\ .$$

Here v_s is the sound-speed, whose value should be given explicitly. [For simplicity the analysis should be carried out at the coordinate origin.]

(d) By introducing the density perturbation $\delta \equiv \Delta/r$ and assuming that **k** is parallel to **V**, show that δ evolves according to

$$\ddot{\delta} + 2(\dot{a}/a)\,\dot{\delta} + (v_s^2k^2 - 4\pi G\rho_1)\,\delta = 0\ .$$

By using the expressions for $(\dot{a}/a)$ and ρ_1 appropriate for an Einstein-de Sitter model with zero pressure and seeking a power-law solution of the form $\delta \propto t^m$, show that the Jeans length is

$$\lambda_J = v_s\,(24\pi/25G\rho_1)^{1/2}$$

and show how δ evolves on scales larger than this.

Question 6

Write an essay on *cosmological nucleosynthesis*. You should indicate the main nuclear reactions involved and include a discussion of the neutron-proton freeze-out ratio and the deuterium bottle-neck effect. You should also explain how the predicted abundances of the relevant elements depend on such factors as the number of neutrino species, the neutron half-life and the baryon density parameter. (If possible, illustrate your discussion with a graph.) Your essay should also include a discussion of the relevant observational results and indicate why cosmological nucleosynthesis is relevant to the dark matter problem.

END OF EXAMINATION

B.J.Carr