QUEEN MARY AND WESTFIELD COLLEGE
UNIVERSITY OF LONDON
M.Sc. Astrophysics

ASTM109/MAS415/MTHMN65 Stellar Structure and Evolution

Wednesday 10th May 2000, 10.00 a.m.

The duration of this examination is three hours.

Answer all parts of section A and no more than two questions from section B. Full marks may be obtained on **complete** *answers to all of section A and one question from section B. More credit will be given for complete answers than a number of fragments.*

The use of a calculator in this examination is permitted, provided that you do not make use of any programming, graph-plotting or algebraic facilities that your calculator may have.

The results of one question may be used without proof in other questions.

The following notation is used throughout unless otherwise stated. The pressure, density, and temperature at an internal point are as usual denoted by P, ρ, and T respectively. The central values of the pressure, density, and temperature are P_c, ρ_c, and T_c respectively. The effective temperature is T_{eff}, the interior luminosity is L_r and M_r is the mass interior to the sphere of radius r. The surface radius, total stellar mass, surface luminosity, and surface gravity are denoted by R, M, L, and g respectively. The opacity, energy generation rate per unit mass, mean molecular weight and ratio of specific heats are κ, ϵ, μ, and γ respectively. $G, \mathcal{R}, h, m_e, m_H, c$, and a denote the gravitational constant, the gas constant, Planck's constant, the mass of the electron, the mass of the hydrogen atom, the velocity of light and the Stefan-Boltzmann radiation constant respectively. The mass fractions of Hydrogen, Helium and heavy elements are X, Y and Z respectively.
You may assume that for a polytrope of index n

$$\rho_c = \frac{3a_n M}{4\pi R^3}, \; P_c = c_n \frac{GM^2}{R^4},$$

where a_n and c_n are constants. In addition if the material is an ideal gas,

$$T_c = b_n \frac{\mu G M}{\mathcal{R} R}, \text{ where } b_n \text{ is a constant.}$$

SECTION A 60 marks

(a) The heavy element content of a sample of stellar material is in the form of ^{12}C. Show that if it is fully ionized the mean molecular weight is given by

$$\mu = \frac{1}{(2X + 3Y/4 + Z/2)}.$$

(b) Starting from the equation of hydrostatic equilibrium, derive the virial theorem for a spherically symmetric star with vanishing surface pressure in the form

$$3\int_0^M \frac{P}{\rho} dM_r = \int_0^M \frac{GM_r}{r} dM_r.$$

(c) A fully convective star is modelled by a polytrope with index $n = 1.5$. It is contracting towards the main sequence at constant T_{eff}. Show that the time required to contract from an initially very large radius is given by

$$t = \frac{GM^2}{7LR}.$$

A group of such stars with different masses has the same age and $T_{eff} = AM^{\beta}$, where A and β are constants. Show that they lie on a line with slope $(4 + 4\beta)/(3\beta)$ in the $(\log L, \log T_{eff})$ plane.

(You may assume without proof that the gravitational energy of a polytrope with index, n, is given by $\Omega = -3GM/((5 - n)R)$, and that the total stellar energy content is one half the gravitational energy .)

(d) For a group of main sequence stars the opacity and energy generation rate are given by $\kappa = \kappa_0 \rho T^{-3.5}$ and $\epsilon = \epsilon_0 \rho T^{14}$, where κ_0, and ϵ_0 are constants. Assuming energy transport is by radiation, derive the two scaling laws for the luminosity in terms of the mass and radius in the form

$$L \propto \frac{M^{5.5}}{R^{0.5}}, \quad L \propto \frac{M^{16}}{R^{17}}.$$

Find the form of the $\log L, \log T_{eff}$ relation for these stars.

(e) The number density of electrons in a completely degenerate non relativistic helium gas occupying the momentum interval $(p, p+dp)$ is given by

$$dn = \frac{8\pi p^2}{h^3}dp, \quad p < p_0, \quad dn = 0, \quad p > p_0.$$

Here p_0 is the Fermi momentum. Show that the density is given by

$$\rho = \frac{16\pi m_H p_0^3}{3h^3}.$$

By performing an appropriate integral over momentum, show that the pressure is given by

$$P = K\rho^{\frac{5}{3}},$$

where K depends only on physical constants. Hence deduce that for non relativistic white dwarfs

$$R = \text{constant} \times M^{-\frac{1}{3}}.$$

(f) In the ideal gas atmosphere of a cool star the opacity is given by $\kappa = \kappa_0 P T^8$, where κ_0 is constant. Given the Eddington approximation for the temperature as a function of the optical depth τ

$$T^4 = T_{eff}^4 \left(\frac{1}{2} + \frac{3\tau}{4}\right),$$

where T_{eff} is the effective temperature, show that for $\gamma = 5/3$, the onset of convection occurs when $\tau = 8/15$.

(You may assume without proof that the onset of convection occurs when $(P/T)dT/dP = (\gamma - 1)/\gamma$.)

(g) The reaction rates R_{ij} are defined such that the number of reactions per unit mass per unit time occuring between particles of species i and j is $R_{ij}X_iX_j$, where X_i and X_j are the mass fractions of species i and j respectively. The energy released per reaction between species i with atomic mass $i(m_H)$ and species j with atomic mass $j(m_H)$ is E_{ij}.

The principal branch of the *pp* chain consists of the reactions

$${}^1H + {}^1H \rightarrow {}^2H + e^+ + \nu, \quad {}^2H + {}^1H \rightarrow {}^3He, \quad {}^3He + {}^3He \rightarrow {}^4He + 2\,{}^1H.$$

Show that

$$\frac{\partial X_1}{\partial t} = m_H(2R_{33}X_3^2 - R_{12}X_1X_2 - 2R_{11}X_1^2), \quad \frac{\partial X_2}{\partial t} = 2m_H(-R_{12}X_1X_2 + R_{11}X_1^2),$$
$$\frac{\partial X_3}{\partial t} = 3m_H(R_{12}X_1X_2 - 2R_{33}X_3^2), \quad \frac{\partial X_4}{\partial t} = 4m_H R_{33}X_3^2.$$

If X_2 and X_3 are in equilibrium, show that the rate of energy production per unit mass is given by

$$\epsilon = (E_{11} + E_{12} + \frac{E_{33}}{2})R_{11}X_{11}^2.$$

(h) Sketch the internal structure of main sequence stars of mass $0.1M_\odot, 1M_\odot$ and $10M_\odot$. In each case indicate which regions are radiative and which convective along with the main sources of opacity and energy generation.

SECTION B Each question 40 marks

B1) An ideal gas convective core of a star is modelled by a polytrope with index $n = 1.5$. Show that if $\rho = \rho_c\theta^{1.5}$, $P = P_c\theta^{2.5}$, and $T = T_c\theta$, θ satisfies

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\theta^{1.5},$$

where the dimensionless radius ξ is defined through

$$r = \alpha\xi,$$

where

$$\alpha = \left(\frac{5\mathcal{R}T_c}{8\pi G\mu\rho_c}\right)^{1/2}.$$

Show that in the neighbourhood of $\xi = 0$,

$$\theta = 1 - \frac{\xi^2}{6} + O(\xi^4).$$

The opacity and energy generation rate are given by $\kappa = \kappa_0\rho/T^s$ and $\epsilon = \epsilon_0\rho T^\beta$ respectively, where s, β, κ_0, and ϵ_0 are constants.

Show that the luminosity is given by

$$L_r = 4\pi\epsilon_0\alpha^3\rho_c^2T_c^\beta\int_0^\xi \xi^2\theta^{(3+\beta)}d\xi$$

and hence

$$L_r = \frac{4\pi\epsilon_0\alpha^3\rho_c^2T_c^\beta}{3}\left(\xi^3 - \xi^5(3+\beta)/10 + O(\xi^7)\right).$$

Show further that the luminosity carried by radiation is given by

$$L_{rad} = -\frac{16\pi ac\alpha T_c^{(4+s)}}{3\kappa_0\rho_c^2}\theta^s\xi^2\frac{d\theta}{d\xi} \quad = \quad \frac{16\pi ac\alpha T_c^{(4+s)}}{3\kappa_0\rho_c^2}\theta^s\int_0^\xi \xi^2\theta^{1.5}d\xi.$$

and hence

$$\frac{L_{rad}}{L_r} = \frac{4acT_c^{(4+s-\beta)}}{3\kappa_0\epsilon_0\alpha^2\rho_c^4}\left(1 - \xi^2(s/6 + 3/20 - (3+\beta)/10) + O(\xi^4)\right).$$

Deduce that the condition for a small convective core is

$$\beta > 5s/3 - 3/2.$$

Use this result, giving expected values for β and s, to acertain whether a convective core is expected for a) stars with $M > 3M_\odot$, b) stars with $M < 1M_\odot$.

B2) The opacity and energy generation rate in massive stars in which radiation pressure may be neglected are given by $\kappa = \kappa_0 Z(1+X)\rho/T^3$, and $\epsilon = \epsilon_0 XZ\rho T^{18}$ respectively. Here κ_0 and ϵ_0 are constants independent of chemical composition. Show that, if energy transport is assumed to be by radiation, then on the main sequence

$$L = C_1 \frac{M^5\mu^7}{Z(1+X)},$$

$$R = C_2 (XZ^2(1+X))^{1/21}\mu^{11/21}M^{5/7}.$$

where the constants C_1, and C_2 are independent of chemical composition.

A star evolves away from the zero age main sequence where the initial luminosity is L_0 and the initial hydrogen mass fraction is X_0 respectively. Assuming that a uniform composition is maintained during the evolution so that X depends only on time t, show that

$$\frac{dX}{dt} = \frac{-L}{EM},$$

where E is the energy liberated on converting unit mass of hydrogen to helium.

Assuming the star has very low metal content so that $\mu = 4/(3+5X)$, show that X is given as a function of time by

$$\frac{1}{9}\left((3+5X_0)^9 - (3+5X)^9\right) + \frac{1}{4}\left((3+5X_0)^8 - (3+5X)^8\right) = \frac{25L_0(1+X_0)(3+5X_0)^7 t}{EM}.$$

At $t = 0$, $X_0 = 0.6$ and the star begins to move off the main sequence when 10% of the hydrogen content is burnt. Show that the main sequence lifetime is

$$t_{ms} = \frac{0.05EM}{L_0}.$$

In a young star cluster a main sequence star with $M = 10M_\odot$ has $L = 10^4 L_\odot$, and $R = 6R_\odot$. The main sequence turn off point occurs at $L = 10^5 L_\odot$. What is the age of the cluster.?

(You may assume $L_\odot/(EM_\odot) = 3.3 \times 10^{-19} sec^{-1}$.)

B3) A model of a red giant consists of a non relativistic helium white dwarf core of mass M_c and radius R_c on top of which there is a radiative envelope of negligible mass which extends to very large radii at low temperature above which there is a convective envelope. The energy production occurs in a thin shell at the base of the radiative envelope. If the opacity is given by $\kappa = \kappa_0 \rho / T^5$, where κ_0 is constant and radiation pressure is negligible, show that in the radiative envelope

$$P = \Lambda T^5,$$

where

$$\Lambda = \left(\frac{16\pi a c G M_c \mu}{15 \kappa_0 L \mathcal{R}} \right)^{1/2}.$$

Deduce that

$$T = \frac{\mu G M_c}{5 \mathcal{R} r}.$$

Show further that the mass in the radiative layers above the shell is finite and given by

$$M_e = 4\pi\Lambda \frac{\mu}{\mathcal{R}} \left(\frac{\mu G M_c}{5\mathcal{R}} \right)^4 \frac{1}{R_c}.$$

The energy generation rate in the shell is given by $\epsilon = \epsilon_0 \rho T^{16}$, where ϵ_0 is constant, show that the luminosity is given by

$$L = 4\pi\epsilon_0 \left(\Lambda \frac{\mu}{\mathcal{R}} \right)^2 \left(\frac{\mu G M_c}{5\mathcal{R}} \right)^{24} \frac{1}{21 R_c^{21}}.$$

Deduce luminosity and envelope mass scalings with core mass in the form

$$L \propto M_c^{16},$$

and

$$M_e \propto M_c^{-19/6}.$$

Early in the red giant phase $M_c = 0.2M$, and the mass in the exterior convective envelope is $0.6M$. What is the mass in the convective envelope after the core mass has increased to $0.3M$?