Queen Mary & Westfield College
UNIVERSITY OF LONDON

MSci

MAS 415 Stellar Structure and Evolution

26th May 1999 10.00. 3hrs

Answer all parts of Section A and no more than 2 questions from Section B. Full marks may be obtained on complete answers to all of section A and one question from section B. More credit will be given for complete answers than for a number of fragments.

The results of one question may be used without proof in other questions.

You are permitted to use an electronic calculator provided that you do not make use of any algebraic, programming, or graph-plotting facilities it may have. Please state on your answer book the name and type of machine used.

The following notation is used in this paper:

M is the total mass, M_r the mass interior to radius r, R is the radius, L the luminosity, T_{eff} the effective temperature and g the surface gravity of a star. P, ρ, T denote the pressure, density and temperature respectively. κ is the opacity per unit mass, ϵ the rate of energy production per unit mass, c_p, c_v the specific heats at constant pressure and volume and $\Gamma = c_p/c_v$. $c. m_e, m_H, m_i, h, G, \mathcal{R}, a, \sigma$ are respectively the velocity of light, the mass of the electron, the mass of the hydrogen atom, the mass of atoms of atomic number i, Planck's constant, the constant of gravity, the gas constant, the Stefan-Boltzmann radiation constant and Stefan's constant. X, Y, Z are the fractions by mass of hydrogen, helium and the heavier elements. $\mathcal{E}_0$ is the energy released by conversion of unit mass of hydrogen to helium. $M_\odot, R_\odot, L_\odot$ are the mass, radius, and luminosity of the Sun.

You may use the following without proof unless specifically asked to derive them:

The radiative energy flux : $F_{rad} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$

The Schwarzschild condition for convection : $\frac{d\log P}{d\log T} \leq \frac{\Gamma}{\Gamma - 1}$

The gravitational energy V, central pressure P_c and central density ρ_c, of a polytrope of index n are:

$V = -\frac{3}{5-n}\frac{GM^2}{R}$, $P_c = a_n \frac{GM^2}{R^4}$, $\rho_c = b_n \frac{M}{R^3}$, where

$a_{1.5} = 0.7701$, $b_{1.5} = 1.4302$: $a_3 = 11.051$, $b_3 = 12.935$: $a_{3.25} = 20.365$, $b_{3.25} = 21.045$

$\frac{GM_\odot^2}{R_\odot L_\odot} = 9.75 \times 10^{14} secs$, $\frac{\mathcal{E}_0 M_\odot}{L_\odot} = 3 \times 10^{18} secs$, $\frac{GM_\odot}{R_\odot^3} = 3.94 \times 10^{-7} secs^{-2}$.

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SECTION A. Answer all parts

a) X is the fraction by mass of hydrogen ^1H, Y that of helium ^4He, and Z that of all other elements. If Z can be approximated by ^{16}O show that the mean molecular weight of a fully ionised mixture is $\mu \approx 4/(3+5X-3Z/4)$.

b) A star is in hydrostatic equilibrium, the equation of state is that of ideal gas $P = (c_p - c_v)\rho T$, $\Gamma = c_p/c_v = 5/3$, and the pressure at the surface can be neglected. If $U = \int c_v T\, dM_r$ is the internal energy and $V = -\int (GM_r dM_r)/r$ the gravitational energy of the star show that $2U + V = 0$

c) If the luminosity of a star of mass M and radius R, composed of an ideal gas, is $L = AM^{4.5}/R^{0.5}$ where A is a constant, and the gravitational energy $V = -3GM^2/2R$, show that the time for a star to contract to radius R from a very large initial radius is $3GM^2/(2RL)$. Show that a group of such stars of the same age and composition, but different masses, lie on a line in the $\log L - \log T_{eff}$ plane, of slope

$$\frac{d\log L}{d\log T_{eff}} = \frac{28}{17}$$

d) In a group of main sequence stars the opacity per unit mass is approximated by $\kappa = \kappa_0 \rho^{0.4} T^{-2}$ where κ_0 is constant, the rate of energy generation per unit mass $\epsilon = \epsilon_0 \rho T^{16}$ where ϵ_0 is constant, and the equation of state is that of a fully ionised ideal gas. Show by order of magnitude arguments, or otherwise, that such stars satisfy the mass-radius, mass-luminosity relations and main sequence relations:

$$\frac{d\log R}{d\log M} = \frac{27}{37}, \quad \frac{d\log L}{d\log M} = \frac{153}{37}, \quad \frac{d\log L}{d\log T_{eff}} = \frac{68}{11}$$

e) Show that in an electron-degenerate ionised gas the electron pressure, in the fully relativistic limit, can be expressed as $P_e = K\rho^{4/3}$, where K is a constant that depends only on the composition. Show that for a given composition a star with this equation of state has a unique mass. (You are not required to evaluate this mass.)

g) The reaction rates R_{ij} are defined such that the number of reactions per unit mass per unit time between particles of species i and j is $R_{ij} X_i X_j$ where X_i, X_j are the fractions by mass of species i and j. E_{ij} is the energy released by the fusion of particle i with particle j. The principal branch of the proton-proton chain is governed by the reactions

$$^1H + {}^1H \rightarrow {}^2H + e^+ + \nu, \quad {}^2H + {}^1H \rightarrow {}^3He, \quad {}^3He + {}^3He \rightarrow {}^4He + 2\,{}^1H$$

Write down the reaction rate equations for this set of reactions. Show that if X_2 and X_3 are in equilibrium, then rate of energy production per unit mass per unit time ϵ, and the equilibrium abundance of ^3He are given by

$$\epsilon = (E_{11} + E_{12} + \frac{1}{2}E_{33}) R_{11} X_1^2, \quad X_{3E} = \left(\frac{R_{11}}{2R_{33}}\right)^{1/2} X_1$$

The reaction rates R_{11} and R_{33} are given by

$$R_{11} = \frac{\rho}{T_6^{2/3}} exp\left(25.4 - \frac{33.8}{T_6^{1/3}}\right), \quad R_{33} = \frac{\rho}{T_6^{2/3}} exp\left(81.1 - \frac{122.8}{T_6^{1/3}}\right)$$

where T_6 is the temperature in units of $10^{6\,o}$K. If the hydrogen abundance $X_1 = 0.7$ show that at a temperature of $13 \times 10^{6\,o}$K, the equilibrium abundance $X_{3E} \approx 6.6 \times 10^{-5}$.

g) In a simple gray model of a stellar atmosphere the variation of temperature T with optical depth τ is given by the Eddington approximation $T^4 = 0.75 T_{eff}^4 (\tau + 2/3)$ where $\tau = -\int \kappa\rho dz$. If the opacity is given by $\kappa = \kappa_0 P T^8$ where P is the pressure and κ_0 a constant, show that the atmosphere is unstable to convection when the temperature $> 0.974 T_{eff}$.

h) Sketch the evolutionary path of a star of solar mass in the Hertzsprung-Russell diagram from pre-main sequence to white dwarf phases. Indicate the regions on this diagram where you expect there to be significant mass loss.

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Section B

B1) Derive the Schwarzschild condition for convective instability in the form

$$\Delta\nabla \equiv \nabla - \nabla_{ad} > 0, \qquad \text{where } \nabla = \frac{d\log T}{d\log P}, \text{ and } \nabla_{ad} = \left(\frac{d\log T}{d\log P}\right)_{ad} = 1 - \frac{1}{\Gamma}$$

Describe the classical mixing length model of convection and show that in this model the convective flux is given by $F_c = f\rho\,(c_p T\Delta\nabla)^{3/2}$ where f is a factor of order unity. Hence deduce that in stellar convective cores $\Delta\nabla << \nabla ad$.

In the stellar cluster M007 stars of different masses have the same age and initial composition (X_0, Z). The opacity per unit mass κ is given by the approximation $\kappa = \kappa_0(1+X)$ where κ_0 is a constant and the equation of state is taken to be that of a fully ionised ideal gas with radiation pressure neglected. A typical main sequence star is observed to have mass of $3M_\odot$, a luminosity of $80L_\odot$ and hydrogen abundance $X_0 = 0.7$. Neglecting the small contribution of Z to the molecular weight show that the luminosity of a star of mass M and hydrogen abundance X is given by

$$\frac{L}{L_\odot} = 2.963\left(\frac{1+X_0}{1+X}\right)\left(\frac{3+5X_0}{3+5X}\right)^4\left(\frac{M}{M_\odot}\right)^3$$

In a simple classical model of stellar evolution these stars have a central convective core which contains a fraction 0.25 of the mass of the star. The stars stay in the main-sequence band until they have consumed the hydrogen in this core and then evolve rapidly into the giant region. If this phase of evolution is modelled by taking the composition to be uniform throughout the star, but changing in time, show that when the star has consumed 25% of its hydrogen its luminosity has increased by a factor of 1.99 (You may neglect the contribution of Z to the molecular weight.)

The cluster M007 has some stars in the giant region and the brightest star in the main sequence band has a luminosity of $10^4 L_\odot$. Assuming this star is just leaving the main sequence band show that its mass is $11.925M_\odot$ and that the age of the cluster is $\approx 2.6\times10^6$ years.

In a model of stellar evolution which includes convective overshooting the core mass is increased by 50% the over the value without overshooting. Using this model of stellar evolution show that the age of the cluster is $\approx 3.9\times10^7$ years.

Note that the values of any constants you may need are given on the front cover of the examination paper.

B2) Show that the electron pressure in a fully degenerate non relativistic helium gas is given by

$$P_e = \frac{8\pi m_e^4 c^5}{15h^3}x^5, \quad \rho = \frac{16\pi c^3 m_e^3 m_H}{3h^3}x^3$$

where all states are filled up to a threshold momentum $p_0 = xm_e c$, and none beyond. Hence show that stars with this equation of state satisfy the mass radius relation $R \propto M^{-1/3}$

The interior of a red giant of total mass M is modelled by a non relativistic degenerate core of mass $M_c = qM$, a thin hydrogen burning shell, a radiative envelope of negligible mass, and a deep outer convective zone; radiation pressure is neglected, the opacity per unit mass κ is constant and the energy generation rate per unit mass is $\epsilon = \epsilon_0 XZ\rho T^{15}$ where ϵ_0 is constant. Show that in the inner regions of the radiative zone, with $M_r \to M_c$

$$\frac{d\log P}{d\log T} = \frac{4}{1 + C/AT^4}, \quad P \to AT^4, \quad T \to \frac{\mu GM_c}{4\mathcal{R}}\frac{1}{r}, \quad \text{where} \quad A = \frac{4\pi acGM_c}{3\kappa L}$$

and C is a constant Assuming the asymptotic envelope solution holds through the hydrogen burning shell, show that the increase of luminosity with increasing core mass satisfies the relation

$$\frac{d\log L}{d\log M_c} = \frac{29}{3}$$

Why do such models of red giants require a deep convective envelope?

The theory of stellar convection gives the entropy parameter $K = P/T^{5/2}$ in the fully ionised interior of convective envelopes of red giants in terms of their surface properties as $K = K_0\ g\ T_{eff}^{-8}$, where K_0 is a constant. If the dimensionless variables x, t, q, E are defined by

$$r = xR, \quad M_r = qM, \quad T(r) = \frac{\mu}{\mathcal{R}}\frac{GM}{R}t(x), \quad E = 4\pi\left(\frac{\mu}{\mathcal{R}}\right)^{5/2} G^{3/2}M^{1/2}R^{3/2}K,$$

show that t, q satisfy the equations

$$\frac{dT}{dx} = -\frac{2}{5}\frac{q}{x^2}, \quad \frac{dq}{dx} = -Ex^2t^{3/2}$$

For $E < E_0$ these equations have the asymptotic behaviour $q \to q_c$ as $x \to 0$. Within the range of q_c appropriate to models of red giants, E varies only slowly; taking the approximation E=constant show that the red giant evolves along a line in the $\log L - \log T_{eff}$ plane with slope

$$\frac{d\log L}{d\log T_{eff}} = -28$$

The hydrogen abundance in the envelope is $X = 0.7$ and when the core mass is $0.3M_\odot$ the luminosity is $20L_\odot$. How long does it take the star to evolve to a luminosity of $10^3 L_\odot$ and what is its core mass at this stage?

What limits this phase of evolution up the giant branch?

Note that the values of any constants you may need are given on the front cover of the examination paper.

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B3) In a simple adiabatic model of the outer convective zones in solar type stars of mass M, luminosity L and radius R, the entropy parameter $K = P/T^{5/2}$ is taken as constant throughout the zone and the opacity is given by the expression $\kappa = \kappa_0 \rho T^{-3}$. Taking the mass of the convective zone to be negligible compared to the mass of the star, and neglecting the small contribution from the surface layers, show that the variation of temperature with radius and the temperature at the base of the zone are given by

$$T(r) = \frac{2}{5}\left(\frac{\mu}{\mathcal{R}}\right) GM \left(\frac{1}{r} - \frac{1}{R}\right), \quad T_b^3 = \frac{15\kappa_0}{32\pi ac}\left(\frac{\mu}{\mathcal{R}}\right) K^2 \frac{L}{GM}$$

The adiabatic sound speed $c(r)$ is defined by $c^2(r) = \Gamma P/\rho$. Taking $\Gamma = 5/3$ show that at the base of the convective zone, the discontinuity in the second derivative of c is given by

$$\left(\frac{d^2c^2}{dr^2}\right)_{rad} - \left(\frac{d^2c^2}{dr^2}\right)_{conv} = -\frac{4}{3}\frac{g^2}{c^2}$$

where $g = GM/r^2$.

This discontinuity gives rise to a periodic signal in the solar oscillation frequencies which indicates that the base of the convective zone lies at an acoustic depth of ≈ 2100 secs, where the acoustic depth

$$\tau = \int_r^R \frac{dr}{c(r)}$$

Using the thin layer approximation $R - r << R$ use the above results to determine the depth of the solar convective zone in units of the solar radius.

Stars of solar type lose angular momentum through a stellar wind at a rate $dJ/dt = -\alpha \dot{M} R_A^2 \Omega$ where α is a constant, $\dot{M}$ is the mass loss rate, Ω the angular velocity and R_A the radius out to which the wind corotates with the star. If $R_A^2 = \beta R^2 \Omega^n$ with n, β constants, and the moment of inertia of the star $J = kMR^2$ then, assuming the star rotates rigidly and neglecting the (very small) change in the star's mass, show that the change in angular velocity with time is given by

$$\frac{d}{dt}\left(\frac{1}{\Omega R^2}\right)^n = n\frac{\alpha\beta}{k}\frac{\dot{M}}{M}\frac{1}{R^{2n}}$$

On the main sequence $\dot{M}$ and R may be taken as constants. Show that $\Omega(t)$ asymptotically satisfies the Skumanich type relation $\Omega \propto t^{-1/n}$.

Observations of the rotation of solar type stars in clusters of different ages suggest that $n = 2$. Assuming this to be the case for the Sun show that parameter combination $\alpha\beta$ is given by

$$\alpha\beta = k_\odot \frac{M_\odot}{\dot{M}_\odot} \frac{1}{2\,\Omega_\odot^2\, t_\odot}$$

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