Queen Mary & Westfield College
UNIVERSITY OF LONDON
MSci EXAMINATION

PHY/945 Physics of Fluids

Time Allowed : 2 Hours.

Answer TWO questions only. No credit will be given for attempting a further question.

Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin. Useful formulae are given at the end of each question.

1

(a) The Euler equation governing the flow of an incompressible ideal fluid in the absence of external forces has the form

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p \quad ,$$

Define what is meant by a streamline. Show that for steady flow the Bernoulli function, $H = \frac{1}{2}u^2 + p/\rho$, is constant along streamlines.

Suppose now that we have compressible ideal fluid in steady isentropic flow. Show that the function $H = \frac{1}{2}u^2 + h$ is constant along streamlines, where h is the enthalpy per unit mass.

For a monatomic ideal gas the enthalpy has the form $h = \frac{5}{2}p/\rho$. In an experiment, this gas, of molar mass M, is stored in a large container at mass density ρ_0, pressure p_0 and temperature T_0. A leak develops in the container wall allowing the gas to escape through a small hole into a vacuum chamber. Show that the gas escapes into the vacuum chamber at a speed **[9]**

$$u_e = \sqrt{\frac{5RT_0}{M}} \quad .$$

NOTE: The following relation holds for any differentiable vector field $\boldsymbol{a}$:

$$(\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{a} = \frac{1}{2}\boldsymbol{\nabla} a^2 + (\boldsymbol{\nabla} \times \boldsymbol{a}) \times \boldsymbol{a} \quad .$$

You may use the thermodynamical relation $dh = Tds + vdp$ where T, s, v and p are respectively temperature, entropy per unit mass, volume per unit mass and pressure.

(b) An ideal gas moves isentropically so that throughout the gas we have $p\rho^{-\gamma} = constant$, where p and ρ are pressure and mass density respectively and γ is the adiabatic constant for the gas.

Write down the equation of continuity expressing mass conservation in the gas flow.

Linearize the mass conservation equation and the Euler equation about an assumed uniform background pressure and density p_0 and ρ_0 by setting $p = p_0 + p_1$ and $\rho = \rho_0 + \rho_1$ with p_1, ρ_1 regarded as small quantities. You may neglect the effect of gravity. Show that the small disturbance of the pressure field, p_1, obeys the three dimensional wave equation

$$\nabla^2 p_1(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2 p_1(\boldsymbol{x}, t)}{\partial t^2} = 0 \quad .$$

Express the phase velocity c of these waves in terms of p_0, ρ_0 and γ. **[11]**

Please see next page

See next page

2

(a) For a given flow field $\boldsymbol{u}$ explain briefly what is meant by (i) vorticity, (ii) circulation, (iii) a vortex line and (iv) a vortex tube.

Two cylindrically symmetric flow fields are given in Cartesian coordinates (with coordinate unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ and position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$) as

$$\boldsymbol{u}_1(\boldsymbol{r}) = \Omega(-y\,\boldsymbol{i} + x\,\boldsymbol{j}) \quad , \quad \boldsymbol{u}_2(\boldsymbol{r}) = \frac{\Gamma}{2\pi(x^2+y^2)}(-y\,\boldsymbol{i} + x\,\boldsymbol{j}) \quad ,$$

where Ω and Γ are constants. For each flow calculate the vorticity.

For ideal incompressible flow, state (without proof) the Kelvin circulation theorem and the two Helmholtz vorticity theorems. **[7]**

(b) The flow $\boldsymbol{u}_2$ above represents a line vortex of strength Γ located on the z - axis. Using the theorems stated above, or otherwise, discuss the motion of two parallel line vortices of equal strength Γ located in unbounded fluid with their centres a distance L apart.

A third parallel line vortex of the same strength Γ is placed at distance L from each of the first two to form an equilateral triangle. Describe the motion of the three vortices and compare it with that of the pair.

Discuss qualitatively why a smoke ring moves. **[8]**

(c) The Navier-Stokes equation for viscous incompressible flow in the absence of external forces has the form

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \mu\nabla^2\boldsymbol{u} \quad .$$

A simple laminar flow field satisfying the Navier-Stokes equation has the form $\boldsymbol{u} = (0, v(x,t), 0)$. Show that the vorticity $\boldsymbol{\omega}$ obeys the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu\nabla^2\boldsymbol{\omega} \quad ,$$

where $\nu = \mu/\rho$.

Explain what this suggests about the way vorticity is transported in a viscous fluid and compare with what happens in an ideal fluid.

[5]

Please turn over

3

(a) The Navier-Stokes equation of motion for an incompressible Newtonian viscous fluid in the absence of external forces has the form

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u} \quad .$$

A certain time-dependent flow has a characteristic length a, a characteristic speed U and a characteristic time T. Using a, U, T, and the mass density ρ, scale the variables appropriately to show that the Navier-Stokes equation may be reduced to the dimensionless form

$$\frac{1}{S}\frac{\partial \boldsymbol{u}'}{\partial t'} + (\boldsymbol{u}' \cdot \boldsymbol{\nabla}')\boldsymbol{u}' = -\boldsymbol{\nabla}' p' + \frac{1}{R}\nabla'^2 \boldsymbol{u}' \quad .$$

Give explicit expressions for the dimensionless numbers S and R in terms of a, U, T, ρ and μ.

Explain briefly how such scaling can be of practical use in the study of fluids. **[8]**

(b) Viscous incompressible fluid flows steadily in the space between two concentric circular cylindrical pipes of radii a_1, a_2 respectively, with $a_2 > a_1$. It is driven by a constant pressure gradient $\boldsymbol{\nabla} p = -\mathcal{P}\boldsymbol{k}$, where $\boldsymbol{k}$ is a unit vector in the z - direction, taken along the common axis of the two pipes. You may assume that the velocity field in cylindrical coordinates has the form $\boldsymbol{u} = w(r)\hat{\boldsymbol{k}}$, where $r = \sqrt{x^2 + y^2}$ is the radial variable.

State the boundary conditions which hold at the surfaces of the inner and outer pipes.

Solve the Navier-Stokes equation for the velocity field $w(r)$, determining unknown constants by use of the boundary conditions above. Show that the volume rate of flow of fluid Q is

$$Q = \frac{\pi \mathcal{P}}{8\mu}\left(a_2^4 - a_1^4 - \frac{(a_2^2 - a_1^2)^2}{\ln(a_2/a_1)}\right) \quad .$$

[12]

NOTE: You may use without proof the result that in cylindrical coordinates for a field of this form we have

$$\nabla^2 \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w(r)}{\partial r}\right)\hat{\boldsymbol{k}} \quad ,$$

and the gradient operator has the form

$$\boldsymbol{\nabla} = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} \quad .$$

In addition

$$\int r \ln r \, dr = \frac{1}{2}r^2 \ln r - \frac{1}{4}r^2 \quad .$$

END OF EXAM - R. B. JONES