4450

Queen Mary and Westfield College

UNIVERSITY OF LONDON

MSci EXAMINATION

PHYSICS OF FLUIDS PHY-945

20 MAY 1998 10:00

Time allowed: TWO HOURS

Answer **TWO** questions only. No credit will be given for attempting a further question.

Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin.

TURN OVER WHEN INSTRUCTED

1

(a) Derive the equation which expresses mass conservation in a fluid in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \quad .$$

If the fluid is incompressible, what condition must $\boldsymbol{u}$ satisfy? [5]

The Euler equation for an ideal fluid of constant density flowing under gravity has the form

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p - \rho \nabla \chi \quad ,$$

where ρ is the uniform mass density and $\chi = gz$ is the gravitational potential energy per unit mass in a coordinate system with $z-$ axis in the vertical direction. Using the Euler equation derive the balance equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \boldsymbol{u}^2\right) = -\nabla \cdot \boldsymbol{e} \quad ,$$

where

$$\boldsymbol{e} = \left(\frac{1}{2}\rho \boldsymbol{u}^2 + \rho\chi + p\right) \boldsymbol{u} \quad .$$

What conservation law is represented by this equation? [5]

(b) In a variation on Newton's rotating bucket experiment, an incompressible fluid completely fills a cylindrical tin of radius a and height h. The tin is sealed top and bottom with horizontal metal plates so that there is no empty space within and no fluid can escape. The container rotates at constant angular velocity Ω about the central axis which is vertical. The fluid inside follows the motion of the container, rotating steadily and rigidly about the same axis and at the same angular velocity Ω.

  i) Write down the form of the velocity field $\boldsymbol{u}$ in a Cartesian coordinate system with $z-$ axis along the axis of rotation. With this given velocity field, solve the steady Euler equation to obtain the pressure field $p(x, y, z)$ throughout the fluid. [7]

  ii) Show that the net force exerted by the rotating fluid on the bottom surface of the container may be expressed as $p_0\pi a^2 + \pi\rho\Omega^2 a^4/4$, where p_0 is a constant. [3]

NOTE: The following relation holds for any differentiable vector field $\boldsymbol{a}$:

$$(\boldsymbol{a} \cdot \nabla)\boldsymbol{a} = \frac{1}{2}\nabla a^2 + (\nabla \times \boldsymbol{a}) \times \boldsymbol{a} \quad .$$

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2

(a) Define what is meant by the vorticity $\boldsymbol{\omega}$ and the circulation associated with a flow field $\boldsymbol{u}$. How are vorticity and circulation related? Give two physical examples where vorticity is important.

[5]

Starting from the Euler equation for incompressible flow under gravity,

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\nabla p - \rho \nabla \chi \quad ,$$

derive the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = 0 \quad .$$

Show that this can be expressed also as

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \boldsymbol{u} \quad .$$

If we have a two dimensional flow field of the form $\boldsymbol{u} = \big(u(x,y,t), v(x,y,t), 0\big)$, show that the equation above becomes

$$\frac{D\boldsymbol{\omega}}{Dt} = 0$$

What is the physical meaning of this result? [7]

(b) A line vortex lying on the $z-$ axis has a flow field given in Cartesian coordinates (with coordinate unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ and position vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$) as

$$\boldsymbol{u}(\boldsymbol{r}) = \frac{\Gamma}{2\pi(x^2+y^2)}(-y\,\boldsymbol{i} + x\,\boldsymbol{j}) \quad ,$$

where Γ is the strength of the vortex. A second line vortex, parallel to the first one and of the same strength Γ is located a distance L away from the first. Describe the subsequent motion of the vortex pair.

If the second vortex has a strength 3Γ instead of Γ, how will the motion be modified?

[8]

NOTE: The following relations hold for any differentiable vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$(\boldsymbol{a} \cdot \nabla)\boldsymbol{a} = \frac{1}{2}\nabla a^2 + (\nabla \times \boldsymbol{a}) \times \boldsymbol{a} \quad .$$

$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) \quad .$$

Please turn over

3

(a) The stress tensor $\boldsymbol{T}$ is a second rank tensor whose components are denoted T_{ij} in a Cartesian coordinate system. Explain what is meant by the component T_{zy}.

State the form of the stress tensor T_{ij} for an incompressible Newtonian viscous fluid. The general equation of motion for a fluid moving under gravity may be written as

$$\rho \frac{Du_i}{Dt} = \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} - \rho \frac{\partial \chi}{\partial x_i} \quad ,$$

where ρ is the mass density and $\chi = gz$ is the gravitational potential energy per unit mass. Give a physical interpretation of the terms in this equation. Use your expression for T_{ij} in an incompressible Newtonian viscous fluid to derive the Navier-Stokes equation of motion

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla} \chi + \mu \nabla^2 \boldsymbol{u} \quad .$$

What are the boundary conditions for this equation and how do they differ from those for an ideal fluid obeying the Euler equation of motion? [8]

(b) A vertical cylindrical pipe of radius a contains viscous incompressible fluid flowing steadily down the pipe under its own weight. The velocity field in cylindrical coordinates has the form

$$\boldsymbol{u} = w(r)\hat{\boldsymbol{k}} \quad ,$$

where $r = \sqrt{x^2 + y^2}$ is the radial variable and $\hat{\boldsymbol{k}}$ is the unit vector in the $z-$ direction taken vertically upwards along the pipe axis. For a field of this form we have

$$\nabla^2 \boldsymbol{u} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w(r)}{\partial r} \right) \hat{\boldsymbol{k}} \quad ,$$

and the gradient operator has the form

$$\boldsymbol{\nabla} = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{k}} \frac{\partial}{\partial z} \quad .$$

Use the steady Navier-Stokes equation to show that the pressure field in the pipe is at most a linear function of z. [6]

Assuming that the pressure is constant throughout the fluid, solve the Navier-Stokes equation for the velocity field $w(r)$. Show that the volume rate of flow of fluid Q is

$$Q = \frac{\pi \rho g a^4}{8\mu} \quad .$$

[6]

END OF EXAM - R. B. JONES