Queen Mary and Westfield College

UNIVERSITY OF LONDON

MSci EXAMINATION

ELECTROMAGNETIC THEORY II PHY-965

22 MAY 1998 14:30

Time allowed: TWO HOURS

Answer **TWO** questions only. No credit will be given for attempting a further question.

Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin.

TURN OVER WHEN INSTRUCTED

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A Formula Sheet is included at the end of the Paper

1 (a) Show that

$$(\nabla^2 + k^2)G_k^{(+)}(\mathbf{x}, \mathbf{x}') = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}'),$$

where

$$G_k^{(+)}(\mathbf{x}, \mathbf{x}') = \frac{e^{ikr}}{r}; \qquad r = |\mathbf{x} - \mathbf{x}'|.$$

[5 marks]

(b) Hence or otherwise derive the equation for the retarded potentials $A^\mu(\mathbf{x}, t)$:

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3\mathbf{x}'\, j^\mu(\mathbf{x}', t - r/c)\frac{1}{r}$$

generated by a localised source j^μ.

[5 marks]

(c) Explain how this equation may be adapted so as to provide a formalism for scattering of radiation from a localised scatterer. Do not attempt to enter into a detailed mathematical description; full credit can be obtained for a clear exposition of the main steps involved, together with an outline of some of the approximations which are often useful. You should make some mention of the following: Born approximation, dipole approximation, differential cross-section.

[10 marks]

2 (a) The Liénard-Wiechert potentials for the electromagnetic fields generated by a charge q following a trajectory $\mathbf{r} = \mathbf{r}(t)$, with instantaneous velocity $\mathbf{u} = \frac{d\mathbf{r}}{dt} = c\boldsymbol{\beta}$, are

$$\Phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}},$$
$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0 q}{4\pi}\left[\frac{\mathbf{u}}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}}.$$

With the aid of a spacetime diagram define the expressions R and $\mathbf{n}$ in these formulae and explain the meaning of the notation $[\cdots]_{\text{ret}}$.

[6 marks]

(b) If $\beta << 1$, at large distances from the charge the electric and magnetic fields are

$$\mathbf{E}_{\text{far}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}{R}\right]_{\text{ret}}, \qquad \mathbf{B}_{\text{far}} = [\mathbf{n}\times\mathbf{E}_{\text{far}}]_{\text{ret}}/c.$$

Show that at large distances, the (Poynting) energy-flux vector is

$$\mathbf{S}_{\text{far}} = \frac{1}{\mu_0 c}|\mathbf{E}_{\text{far}}|^2\mathbf{n}.$$

[2 marks]

(c) Derive the Larmor formula

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}|\dot{\mathbf{u}}|^2$$

for the total instantaneous power radiated by a non-relativistic accelerated charge.

[4 marks]

(d) Estimate how long it would take an electron in a hydrogen atom to spiral into the nucleus according to classical electrodynamics, stating clearly the assumptions you make. [An order-of-magnitude calculation will suffice. You may take the initial radius of the electron's orbit to be $r_0 = 5.3 \times 10^{-11}$m, and the initial speed to be $c\alpha$, $\alpha = 1/137$.]

[8 marks]

3 (a) Unpolarised electromagnetic radiation with wave-vector $\mathbf{k} = k(1,1,1)/\sqrt{3}$ is incident on a small dielectric sphere of radius ρ, with $k\rho >> 1$.

Using the dipole approximation and the formula

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\epsilon}; \mathbf{n}_0, \boldsymbol{\epsilon}_0) = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2\frac{1}{E_0^2}|\boldsymbol{\epsilon}^*\cdot\mathbf{p}|^2,$$

(where the symbols have their usual meaning), find the differential scattering cross-section in the direction of the positive x-axis.
You may also use the relation

$$\mathbf{p} = 4\pi\epsilon_0\rho^3\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\mathbf{E}$$

for the induced dipole moment $\mathbf{p}$ on a dielectric sphere of radius ρ immersed in an electric field $\mathbf{E}$.

[12 marks]

(b) How is this result modified if the single scatterer is replaced by a collection of identical scatterers, all occupying a region small compared with the wave-length of the radiation.

[3 marks]

(c) The single sphere of the first part of the Question is replaced by an array of six identical spheres, each of radius ρ, placed at the positions

$$(\pm a, 0, 0),\ (0, \pm a, 0)\ (0, 0, \pm a); \qquad \rho << a.$$

What is the differential scattering cross-section for this array in the direction of the positive x-axis?

[5 marks]

FORMULA SHEET

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

$$\begin{aligned}\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a},\\ \nabla\times(\psi\mathbf{a}) &= (\nabla\psi)\times\mathbf{a} + \psi(\nabla\times\mathbf{a}),\\ \nabla\times(\nabla\times\mathbf{a}) &= \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}.\end{aligned}$$

$$\nabla\big(\psi(r)\big) = \mathbf{n}\psi'(r).$$

Maxwell's equations:

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t};$$
$$\nabla\cdot\mathbf{D} = \rho, \qquad \nabla\times\mathbf{H} = \mathbf{j} + \frac{\partial\mathbf{D}}{\partial t}.$$

For linear isotropic media:

$$\begin{aligned}\mathbf{D} &= \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P},\\ \mathbf{H} &= \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.\end{aligned}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

The "standard boost":

$$\begin{aligned}x'^0 &= \gamma(x^0 - \beta x^1)\\ x'^1 &= \gamma(x^1 - \beta x^0)\\ x'^2 &= x^2\\ x'^3 &= x^3,\end{aligned}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

$$\begin{aligned}c^2 d\tau^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2\\ &= dx^\alpha \eta_{\alpha\beta} dx^\beta.\end{aligned}$$

$$\eta_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta = 0\\ \text{-1} & \text{if } \alpha = \beta = 1,2,3\\ 0 & \text{if } \alpha \neq \beta\end{cases}$$

$$\begin{aligned}\partial_\mu = \frac{\partial}{\partial x^\mu} &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)\\ \partial^\mu &= \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right).\end{aligned}$$

$$\Box = \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

$$A^\alpha = (A^0, \mathbf{A}) = \left(\frac{1}{c}\Phi, \mathbf{A}\right) \qquad A_\alpha = \left(\frac{1}{c}\Phi, -\mathbf{A}\right).$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}.$$

$$j^\alpha = (j^0, \mathbf{j}) = (c\rho, \mathbf{j}).$$

$$\Box A^\alpha - \partial^\alpha(\partial_\beta A^\beta) = \mu_0 j^\alpha.$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha;$$

$$\|F^{\alpha\beta}\| = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

Maxwell's equations again:

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta;$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$

End of paper