2pt

Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-966(4261) Electromagnetic Theory

Time Allowed: 2 hours 30 minutes

Date: 1st May 2002

Time: 10:00

Instructions: **Answer THREE QUESTIONS only. Each question carries 20 marks. An indicative marking-scheme is shown in square brackets [ ] after each part of a question. A formula sheet is provided at the end of the examination paper.**

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1. The symmetric stress-energy-momentum tensor $\Theta^{\alpha\beta}$ is defined by

$$\Theta^{\alpha\beta} = \frac{1}{\mu_0}[F^{\alpha\lambda}F_\lambda{}^\beta - \frac{1}{4}\eta^{\alpha\beta}F^{\mu\lambda}F_{\lambda\mu}].$$

(a) Show that

$$\Theta^{00} = \frac{1}{2}\epsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2),$$

and

$$\Theta^{0i} = \frac{1}{c\mu_0}(\mathbf{E}\times\mathbf{B})^i,$$

and give the physical significance of these quantities. [8 marks]

(b) Prove

$$\partial_\alpha\Theta^{\alpha\beta} = j_\alpha F^{\alpha\beta}.$$

[6 marks]

(c) For $\beta = 0$ show that this becomes Poynting's equation

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{j}.$$

[2 marks]

(d) What is the physical meaning of this equation? [4 marks]

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2. In the dipole approximation for a scattering centre at the origin, the electric and magnetic fields for the scattered radiation are given by

$$\mathbf{E}_{\rm SC} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\Big((\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c\Big),$$

$$\mathbf{B}_{\rm SC} = \mathbf{n}\times\mathbf{E}_{\rm SC}/c,$$

where $\mathbf{p}$ and $\mathbf{m}$ are the induced dipole electric and magnetic dipole moments of the scatterer. If the incident wave is a plane wave given by

$$\mathbf{E}_{\rm in} = \mathbf{E}_0 e^{i\mathbf{k}_0\cdot\mathbf{x}}, \qquad \mathbf{B}_{\rm in} = \mathbf{n}_0\times\mathbf{E}_{\rm in}/c,$$

with $\mathbf{k}_0 = k\mathbf{n}_0$, the differential scattering cross-section may be written as

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \frac{r^2\langle|\mathbf{S}_{\rm SC}|\rangle}{\langle|\mathbf{S}_{\rm in}|\rangle},$$

where $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ is the Poynting flux vector and the notation $\langle\cdots\rangle$ indicates time-averaging.

(a) Show that this reduces to

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \Big(\frac{k^2}{4\pi\epsilon_0}\Big)^2\frac{1}{E_0^2}\Big((\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c\Big)^2.$$

[5 marks]

(b) Now consider a collection of identical dipole scattering centres, located at the points $\mathbf{x}_j$. Show that the effect is to multiply the cross-section for a single scatterer by the structure factor

$$\mathcal{F}(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2,$$

[5 marks]

where $\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n})$.

(c) Show that for N scatterers $\mathcal{F}(0) = N^2$, and find an approximation for $\mathcal{F}(\mathbf{q})$ for $N >> 1$ scatterers distributed at random, with a a typical distance apart, for $|\mathbf{q}|a >> 1$. [5 marks]

(d) Explain what happens if the scatterers are spaced regularly, as for example in a crystal. [5 marks]

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3. The vector potential $\mathbf{A}(\mathbf{x})e^{-i\omega t}$ far from an oscillating magnetic dipole $\mathbf{m}e^{-i\omega t}$ at the origin is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r} \mathbf{n} \times \mathbf{m}.$$

(a) Define k, r and $\mathbf{n}$ in this equation. [3 marks]

(b) What is the magnetic field $\mathbf{B}$ far from the oscillating dipole? [5 marks]

(c) The Poynting vector $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{c}{\mu_0} |\mathbf{B}|^2 \mathbf{n}.$$

Show that this reduces to

$$\mathbf{S} = \frac{\mu_0}{16\pi^2} \frac{\omega^4}{c^3} \frac{1}{r^2} |\mathbf{m}|^2 \sin^2\theta \, \mathbf{n}.$$

What is the angle θ in this expression? [5 marks]

(d) A neutron star rotates with angular rotation frequency ω. It has a magnetic dipole moment of magnitude m, but this is misaligned with the axis of rotation by a constant angle α. Show that it radiates energy at a rate

$$\frac{dE}{dt} = -\frac{\mu_0}{6\pi} \frac{\omega^4}{c^3} m^2 \sin^2\alpha.$$

[7 marks]

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4. (a) Consider spherical coordinates (r, θ, ϕ), and a vector field $\mathbf{A}$ with components

$$A_r = 0, \quad A_\theta = 0, \quad A_\phi = \frac{g(1 - \cos\theta)}{4\pi r \sin\theta}.$$

Show that this potential gives a magnetic field

$$\mathbf{B} = \frac{g}{4\pi r^2}\,\hat{\mathbf{r}}.$$

[5 marks]

(b) Explain what this magnetic field describes. [3 marks]

(c) The Lorentz force law for the motion in this field of a particle of electric charge e, rest mass m, velocity $\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ and momentum $\mathbf{p} = \gamma(v)m\mathbf{v}$ gives

$$\dot{\mathbf{p}} = \frac{eg}{4\pi}\frac{\mathbf{v} \times \mathbf{r}}{r^3}.$$

Show that the quantities

$$E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}, \qquad \mathbf{J} = \gamma(v)m\mathbf{r} \times \mathbf{v} - \frac{eg}{4\pi}\frac{\mathbf{r}}{r},$$

are constants of the motion and explain what these invariants are physically and what the separate terms in $\mathbf{J}$ represent. [8 marks]

(d) Consider the case where the particle in part (c) above is stationary. Assuming that $\mathbf{J}$ has the properties of intrinsic angular momentum, derive the quantisation condition

$$\frac{eg}{4\pi} = \frac{n}{2}\hbar, \qquad n = 0, \pm 1, \pm 2, \ldots$$

[4 marks]

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5. (a) Show that Maxwell's equations imply that the current is conserved, $\partial_\beta j^\beta = 0$. [4 marks]

(b) Consider the gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha \Lambda$, for some function $\Lambda(x)$. Explain how the Lagrangian

$$L = \int d^4x \Big(-\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - A_\alpha j^\alpha \Big)$$

is invariant under this transformation. [5 marks]

(c) Write down the equation for the Lorentz force on a particle of charge q moving with velocity $\mathbf{v}$. Explain how this may be generalised to the equation

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B}$$

for the force per unit volume on a charge and current density. [5 marks]

(d) Show that

$$f^k = F^{k\alpha} j_\alpha, \qquad \text{for } k = 1, 2, 3,$$

and define f^0 such that f^α is a four vector. [6 marks]

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Formula Sheet

$$
\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c},\\
\nabla\cdot(\psi\mathbf{a}) &= \mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a},\\
\nabla\times(\psi\mathbf{a}) &= (\nabla\psi)\times\mathbf{a}+\psi(\nabla\times\mathbf{a}),\\
\nabla\times(\nabla\times\mathbf{a}) &= \nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a},\\
\nabla(\psi(r)) &= \mathbf{n}\psi'(r).
\end{aligned}
$$

Maxwell's equations:

$$
\begin{aligned}
\nabla\cdot\mathbf{B}=0, &\qquad \nabla\times\mathbf{E}=-\tfrac{\partial\mathbf{B}}{\partial t};\\
\nabla\cdot\mathbf{D}=\rho, &\qquad \nabla\times\mathbf{H}=\mathbf{j}+\tfrac{\partial\mathbf{D}}{\partial t}.
\end{aligned}
$$

$$\mathbf{F}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B}).$$

For linear isotropic media:

$$\mathbf{D}=\epsilon\mathbf{E}=\epsilon_0\mathbf{E}+\mathbf{P},\qquad \mathbf{H}=\frac{1}{\mu}\mathbf{B}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M}.$$

$$c^2d\tau^2=c^2dt^2-dx^2-dy^2-dz^2=dx^\alpha\eta_{\alpha\beta}dx^\beta.$$

$$\eta_{\alpha\beta}=\begin{cases}+1 & \text{if } \alpha=\beta=0\\ -1 & \text{if } \alpha=\beta=1,2,3\\ 0 & \text{if } \alpha\neq\beta\end{cases}$$

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\Big(\frac{1}{c}\frac{\partial}{\partial t},\nabla\Big),\qquad \partial^\mu=\Big(\frac{1}{c}\frac{\partial}{\partial t},-\nabla\Big).$$

$$\partial_\alpha F^{\alpha\beta}=\partial_\alpha\partial^\alpha A^\beta-\partial^\beta\partial_\alpha A^\alpha=\mu_0 j^\alpha;\qquad F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha.$$

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0.$$

$$\|F^{\alpha\beta}\|=\begin{pmatrix}0 & -E^1/c & -E^2/c & -E^3/c\\ E^1/c & 0 & -B^3 & B^2\\ E^2/c & B^3 & 0 & -B^1\\ E^3/c & -B^2 & B^1 & 0\end{pmatrix}.$$

In spherical coordinates (r,θ,ϕ), for a vector field $\mathbf{A}$ with components (A_r,A_θ,A_ϕ),

$$
\begin{aligned}
\nabla\times\mathbf{A}= \;& \hat{\mathbf{r}}\,\tfrac{1}{r\sin\theta}\Big(\tfrac{\partial}{\partial\theta}(A_\phi\sin\theta)-\tfrac{\partial A_\theta}{\partial\phi}\Big)+\hat{\theta}\Big(\tfrac{1}{r\sin\theta}\tfrac{\partial A_r}{\partial\phi}-\tfrac{1}{r}\tfrac{\partial}{\partial r}(rA_\phi)\Big)\\
&+\hat{\phi}\tfrac{1}{r}\Big(\tfrac{\partial}{\partial r}(rA_\theta)-\tfrac{\partial A_r}{\partial\theta}\Big)
\end{aligned}
$$

End of Examination Paper **Dr WJ Spence**