Queen Mary
UNIVERSITY OF LONDON
MSci EXAMINATION

MSci 4261, QMW/PHY 966 Electromagnetic Theory

Time Allowed : 2 hours 30 minutes.

Answer THREE questions only. All questions carry equal marks.

A FORMULA SHEET is provided at the end of this paper.

Marks shown are indicative of those the Examiners anticipate assigning.

1 The symmetric stress-energy-momentum tensor $\Theta^{\alpha\beta}$ is defined by

$$\Theta^{\alpha\beta} = \frac{1}{\mu_0}[F^{\alpha\lambda}F_\lambda{}^\beta - \frac{1}{4}\eta^{\alpha\beta}F^{\mu\lambda}F_{\lambda\mu}].$$

(a) Show that

$$\Theta^{00} = \frac{1}{2}\epsilon_0(\mathbf{E}^2 + c^2\mathbf{B}^2),$$

and

$$\Theta^{0i} = \frac{1}{c\mu_0}(\mathbf{E} \times \mathbf{B})^i,$$

and give the physical significance of these quantities. [8 marks]

(b) Prove

$$\partial_\alpha \Theta^{\alpha\beta} = j_\alpha F^{\alpha\beta}.$$ [6 marks]

(c) For $\beta = 0$ show that this becomes Poynting's equation

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}.$$ [2 marks]

(d) What is the physical meaning of this equation? [4 marks]

2 In the dipole approximation for a scattering centre at the origin, the electric and magnetic fields for the scattered radiation are given by

$$\mathbf{E}_{SC} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c],$$
$$\mathbf{B}_{SC} = \mathbf{n}\times\mathbf{E}_{SC}/c;$$

where $\mathbf{p}$ and $\mathbf{m}$ are the induced dipole electric and magnetic dipole moments of the scatterer. If the incident wave is a plane wave given by

$$\mathbf{E}_{in} = \mathbf{E}_0 e^{i\mathbf{k}_0\cdot\mathbf{x}},$$
$$\mathbf{B}_{in} = \mathbf{n}_0\times\mathbf{E}_{in}/c,$$

with $\mathbf{k}_0 = k\mathbf{n}_0$, the differential scattering cross-section may be written as

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \frac{r^2\langle|\mathbf{S}_{SC}|\rangle}{\langle|\mathbf{S}_{in}|\rangle},$$

where $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ is the Poynting flux vector and the notation $\langle\cdots\rangle$ indicates time-averaging.

(a) Show that this reduces to

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\mathbf{n}_0) = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2\frac{1}{E_0^2}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} + \mathbf{m}\times\mathbf{n}/c]^2.$$ [5 marks]

(b) Now consider a collection of identical dipole scattering centres, located at the points $\mathbf{x}_j$. Show that the effect is to multiply the cross-section for a single scatterer by the structure factor

$$\mathcal{F}(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2,$$ [5 marks]

where $\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n})$.

(c) Show that for N scatterers $\mathcal{F}(0) = N^2$, and find an approximation for $\mathcal{F}(\mathbf{q})$ for $N >> 1$ scatterers distributed at random, with a a typical distance apart, for $|\mathbf{q}|a >> 1$. [5 marks]

(d) Explain what happens if the scatterers are spaced regularly, as for example in a crystal. [5 marks]

3 The electric dipole contribution to the vector potential is given by

$$\mathbf{A} = -\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{ik}{c}\mathbf{p}.$$

(a) Define k and r in this equation. [2 marks]

(b) What is the magnetic field $\mathbf{B}$? [4 marks]

(c) Show that in the far zone $kr >> 1$

$$\mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\frac{k^2}{c}\mathbf{n}\times\mathbf{p},$$
$$\mathbf{E} = c\mathbf{B}\times\mathbf{n}$$

[8 marks]

(d) From the expression

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0 c}|r\mathbf{E}|^2$$

for the power radiated per unit solid angle, show that the total power radiated in all directions by pure electric dipole radiation is equal to

$$\frac{\mu_0}{12\pi c}|\mathbf{p}|^2\omega^4$$

[6 marks]

4 The Liénard-Wiechert potentials for the electromagnetic fields generated by a charge q following a trajectory $\mathbf{r} = \mathbf{r}(t)$, with instantaneous velocity $\mathbf{u} = \frac{d\mathbf{r}}{dt} = c\boldsymbol{\beta}$, are

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}},$$
$$\mathbf{A} = \frac{\mu_0 qc}{4\pi}\left[\frac{\boldsymbol{\beta}}{R}\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right]_{\text{ret}}.$$

(a) Explain the meaning of the notation $[\ldots]_{\text{ret}}$, and define the distance R and the direction vector $\mathbf{n}$. [4 marks]

(b) If $|\boldsymbol{\beta}| << 1$, show that at large distances from the charge the electric field is

$$\mathbf{E}_{\mathbf{far}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{1}{R}(\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}}))\right]_{\text{ret}}.$$ [6 marks]

(c) Assuming that the corresponding magnetic field is given by

$$\mathbf{B}_{\mathbf{far}} = \left[\mathbf{n}\times\mathbf{E}_{\mathbf{far}}\right]_{\mathbf{far}}/c,$$

show that at large distances, the Poynting energy-flux vector is

$$\mathbf{S}_{\mathbf{far}} = \frac{1}{\mu_o c}|\mathbf{E}_{\mathbf{far}}|^2\mathbf{n}.$$ [4 marks]

(d) Derive the Larmor formula

$$P = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}|\dot{\mathbf{u}}|^2$$

for the total instantaneous power radiated by a non-relativistic accelerated charge. [6 marks]

5 (a) Show that Maxwell's equations imply that the current is conserved, $\partial_\beta j^\beta = 0$. [4 marks]

(b) Consider the gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha\Lambda$, for some function $\Lambda(x)$. Explain how the Lagrangian

$$L = \int d^4x\left(-\frac{1}{4\mu_0}F^{\alpha\beta}F_{\alpha\beta} - A_\alpha j^\alpha\right)$$

is invariant under this transformation. [5 marks]

(c) Write down the equation for the Lorentz force on a particle of charge q moving with velocity $\mathbf{v}$. Explain how this may be generalised to the equation

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}$$

for the force per unit volume on a charge and current density. [5 marks]

(d) Show that

$$f^k = F^{k\alpha}j_\alpha, \qquad \text{for } k = 1,2,3,$$

and define f^0 such that f^α is a four vector. [6 marks]

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \boldsymbol{\nabla}\psi + \psi \boldsymbol{\nabla} \cdot \mathbf{a}, \\ \boldsymbol{\nabla} \times (\psi \mathbf{a}) &= (\boldsymbol{\nabla}\psi) \times \mathbf{a} + \psi(\boldsymbol{\nabla} \times \mathbf{a}), \\ \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2 \mathbf{a}. \end{aligned}$$

$$\boldsymbol{\nabla}\big(\psi(r)\big) = \mathbf{n}\psi'(r).$$

Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t};$$
$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho, \qquad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}.$$

For linear isotropic media:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P},$$
$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

The "standard boost":

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3, \end{aligned}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ &= dx^\alpha \eta_{\alpha\beta} dx^\beta. \end{aligned}$$

$$\eta_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta = 0 \\ \text{-1} & \text{if } \alpha = \beta = 1,2,3 \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\begin{aligned} \partial_\mu = \frac{\partial}{\partial x^\mu} &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \\ \partial^\mu &= \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right). \end{aligned}$$

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

$$A^\alpha = (A^0, \mathbf{A}) = \left(\frac{1}{c}\Phi, \mathbf{A}\right) \qquad A_\alpha = \left(\frac{1}{c}\Phi, -\mathbf{A}\right).$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad \mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial}{\partial t}\mathbf{A}.$$

$$j^\alpha = (j^0, \mathbf{j}) = (c\rho, \mathbf{j}).$$

$$\Box A^\alpha - \partial^\alpha(\partial_\beta A^\beta) = \mu_0 j^\alpha.$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha;$$

$$\|F^{\alpha\beta}\| = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

Maxwell's equations again:

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta;$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$