# Queen Mary & Westfield College

## UNIVERSITY OF LONDON

## MSci EXAMINATION

**PHY/966 Electromagnetic Theory**

**Time Allowed : 2 hours 30 minutes.**

Date: 24th May 1999 Time: 10.00

**Answer THREE questions only. All questions carry equal marks.**

**A FORMULA SHEET is provided at the end of this paper**

1 The vector potential $\mathbf{A}(\mathbf{x})e^{-i\omega t}$ far from an oscillating magnetic dipole $\mathbf{m}e^{-i\omega t}$ at the origin is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} ik \frac{e^{ikr}}{r} \mathbf{n} \times \mathbf{m}.$$

(a) Define $k$, $r$ and $\mathbf{n}$ in this equation

[3marks]

(b) What is the magnetic field B far from the oscillating dipole?

[5 marks]

(c) The Poynting vector $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{c}{\mu_0} |\mathbf{B}|^2 \mathbf{n}.$$

Show that this reduces to

$$\mathbf{S} = \frac{\mu_0}{16\pi^2} \frac{\omega^4}{c^3} \frac{1}{r^2} |\mathbf{m}|^2 \sin^2\theta \, \mathbf{n}.$$

What is the angle $\theta$ in this expression?

[5 marks]

(d) A neutron star rotates with angular rotation frequency $\omega$. It has a magnetic dipole moment of magnitude $m$, but this is misaligned with the axis of rotation by a constant angle $\alpha$. Show that it radiates energy at a rate

$$\frac{dE}{dt} = -\frac{\mu_0}{6\pi} \frac{\omega^4}{c^3} m^2 \sin^2\alpha.$$

[7 marks]

**2** (a) Write down the equation for the Lorentz force on a particle of charge $q$ moving with velocity $\mathbf{v}$. Explain how this may be generalised to the equation

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}$$

for the force per unit volume on a charge and current density.

[5 marks]

(b) Given that the components of the (skew-symmetric) field-strength tensor $F^{\mu\nu}$ are

$$F^{0i} = -E^i/c \qquad \text{for } i = 1, 2, 3$$

$$F^{12} = -B^3 \quad F^{23} = -B^1 \quad F^{31} = -B^2,$$

show that

$$f^k = F^{k\lambda} j_\lambda \qquad \text{for } k = 1, 2, 3.$$

[5 marks]

(c) Define $f^0$ such that $f^\mu$ is a 4-vector, and then show that

$$f^0 = \frac{1}{c} \times (\text{rate of change of energy of charged matter per unit volume}),$$

and write a similar equation for $\mathbf{f}$.

[10 marks]

**3** An incident electromagnetic field

$$\mathbf{E}_{\mathrm{INC}} = \mathbf{E}_0 e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega t)}$$

$$\mathbf{B}_{\mathrm{INC}} = \frac{1}{c}\mathbf{n}_0 \times \mathbf{E}_{\mathrm{INC}}$$

(where $\mathbf{k}_0 = k\mathbf{n}_0$) induces an oscillating electric dipole moment $\mathbf{p}e^{-i\omega t}$ on a dielectric scatterer which in turn generates a scattered electromagnetic field; in the far region this is given by

$$\mathbf{E}_{\mathrm{SC}} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}(\mathbf{n} \times \mathbf{p}) \times \mathbf{n}$$

$$\mathbf{B}_{\mathrm{SC}} = \frac{1}{c}\mathbf{n} \times \mathbf{E}_{\mathrm{SC}}.$$

Here the field point is $\mathbf{x} = r\mathbf{n}$, and the scatterer is at the origin.

(a) Derive the formula

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\epsilon}) = \left(\frac{k^2}{4\pi\epsilon_0}\right)^2 \frac{1}{E_0^2}|\boldsymbol{\epsilon}^* \cdot \mathbf{p}|^2$$

for the differential cross-section for scattering in the direction $\mathbf{n}$ with polarisation vector $\boldsymbol{\epsilon}$.

[10 marks]

(b) Show that for a collection of identical scatterers of this kind, the corresponding differential cross-section is as above, but with an additional factor, the *structure factor* given by

$$\mathcal{F}(\mathbf{q}) = |\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}|^2,$$

where $\mathbf{q} = k(\mathbf{n}_0 - \mathbf{n})$.

[10 marks]

4 (a) Show that the equations for the Lorentz transformation may be written as

$$x'^0 = \gamma(x^0 - \boldsymbol{\beta}\cdot\mathbf{x})$$
$$\mathbf{x}' = \Big(\mathbf{x} - \frac{\boldsymbol{\beta}\cdot\mathbf{x}\,\boldsymbol{\beta}}{\beta^2}\Big) + \boldsymbol{\beta}\gamma\Big(\frac{\boldsymbol{\beta}\cdot\mathbf{x}}{\beta^2} - x^0\Big),$$

where the symbols have their usual significance.

[8 marks]

(b) The frame $K'$ is moving with velocity $\mathbf{v}_1$ relative to $K$, and $K''$ is moving with velocity $\mathbf{v}_2$ relative to $K'$. Find the velocity $\mathbf{v}$ of $K''$ relative to $K$. You may *not* assume that all velocities are along the $x$-axis!

[12 marks]

5 Consider a dielectric sphere with relative permittivity $\epsilon_r$ and radius $a$ placed in an electric field which is initially uniform, of magnitude $E_0$ and directed along the $z$-axis.

(a) With the $z$-axis as polar axis, verify that the electrostatic potential $\Phi$ may be written as

$$\Phi_{\text{in}} = -\Big(\frac{3}{\epsilon_r+2}\Big)E_0 r\cos\theta$$

inside the sphere, and

$$\Phi_{\text{out}} = -E_0 r\cos\theta + \Big(\frac{\epsilon_r-1}{\epsilon_r+2}\Big)E_0\frac{a^3}{r^2}\cos\theta$$

outside the sphere.

[5 marks]

(b) Show that inside the sphere there is a constant $E$-field parallel to the applied field, and using the definition $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ show that the polarisation $\mathbf{P}$ is

$$\mathbf{P} = 3\epsilon_0\Big(\frac{\epsilon_r-1}{\epsilon_r+2}\Big)\mathbf{E}_0.$$

Hence determine the total induced dipole moment $\mathbf{p}$ of the sphere.

[5 marks]

(c) The formula for the differential scattering cross-section from an electric dipole is

$$\frac{d\sigma}{d\Omega} = \Big(\frac{k^2}{4\pi\epsilon_0}\Big)^2\frac{1}{E_0^2}|\boldsymbol{\epsilon}^*\cdot\mathbf{p}|^2.$$

Use this to obtain an expression for the differential scattering cross-section for light scattered from the sphere previously considered.

[5 marks]

(d) Sunlight scattered from a clear sky appears blue, and is polarised. Give an explanation for this based on the results you have just obtained.

[5 marks]

You may wish to use the expression for $\nabla$ and $\nabla^2$ in spherical polar coordinates:

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi},$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big) + \frac{1}{r^2\sin^2\theta}\Big(\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

## FORMULA SHEET

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

$$\begin{aligned} \nabla \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \nabla\psi + \psi \nabla \cdot \mathbf{a}, \\ \nabla \times (\psi \mathbf{a}) &= (\nabla\psi) \times \mathbf{a} + \psi(\nabla \times \mathbf{a}), \\ \nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}. \end{aligned}$$

$$\nabla\big(\psi(r)\big) = \mathbf{n}\psi'(r).$$

Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t};$$

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}.$$

For linear isotropic media:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P},$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

The "standard boost":

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3, \end{aligned}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ &= dx^\alpha \eta_{\alpha\beta} dx^\beta. \end{aligned}$$

$$\eta_{\alpha\beta} = \begin{cases} +1 & \text{if } \alpha = \beta = 0 \\ \text{-1} & \text{if } \alpha = \beta = 1,2,3 \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\begin{aligned} \partial_\mu = \frac{\partial}{\partial x^\mu} &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \\ \partial^\mu &= \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right). \end{aligned}$$

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

$$A^\alpha = (A^0, \mathbf{A}) = \left(\frac{1}{c}\Phi, \mathbf{A}\right) \qquad A_\alpha = \left(\frac{1}{c}\Phi, -\mathbf{A}\right).$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A}.$$

$$j^\alpha = (j^0, \mathbf{j}) = (c\rho, \mathbf{j}).$$

$$\Box A^\alpha - \partial^\alpha(\partial_\beta A^\beta) = \mu_0 j^\alpha.$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha;$$

$$\|F^{\alpha\beta}\| = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

Maxwell's equations again:

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta;$$
$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0.$$

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