UNIVERSITY OF LONDON

MSci EXAMINATION 2002

For Internal Students of

Royal Holloway

DO NOT TURN OVER UNTIL TOLD TO BEGIN

PH4241A: RELATIVISTIC QUANTUM MECHANICS

Time Allowed: TWO AND A HALF hours

Answer THREE QUESTIONS only

No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

Only CASIO fx85WA Calculators are permitted

PH4241/28

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	H m^{-1}
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	F m^{-1}
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	m F^{-1}
Speed of light in vacuum	c	=	3.00×10^{8}	m s^{-1}
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	C kg^{-1}
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	W m^{-2} K^{-4}
Gas constant	R	=	8.31	J mol^{-1} K^{-1}
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	N m^2 kg^{-2}
Acceleration due to gravity	g	=	9.81	m s^{-2}
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^3
One standard atmosphere	P_0	=	1.01×10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

$e \cong 2.718$ $\pi \cong 3.142$ $\log_e 10 \cong 2.303$

PART MARKS

1. (a) Discuss what is meant by a conserved quantity in quantum mechanics. **[2]**

If $\hat{L}_z$ is the z-component of the orbital angular momentum operator for a one-particle system and $\hat{F}$ is any other operator show that

$$\left[\hat{L}_z , \hat{\mathrm{F}}\right] = -i\hbar\frac{\partial\hat{F}}{\partial\phi}$$

where (r, θ, ϕ) are spherical polar coordinates. **[3]**

Hence, show that invariance of the Hamiltonian for a one-particle system under rotations about the z-axis implies conservation of the z-component of orbital angular momentum. **[6]**

Generalise the discussion to show that this result is still correct for an N-particle system, for arbitrary N. **[6]**

For this part, you may ***assume*** that for any operator $\hat{F}$ for an N-Particle system

$$[\hat{L}_z, \hat{F}] = -i\hbar \sum_{i=1}^{N} \frac{\partial\hat{F}}{\partial\phi_i}.$$

Hence, show that the z-component of orbital angular momentum is conserved for an atom with Coulomb forces between the electrons and between the electrons and a static nucleus. **[3]**

You may ***assume*** that ∇^2 is a scalar under rotations.

2. Using the representation for the Dirac matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \ \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \ i = 1, 2, 3,$$

Construct the general positive energy plane wave solution of the free particle Dirac Equation, for non-zero momentum. **[8]**

Also construct the general negative energy plane wave solution, for non-zero momentum. **[5]**

Using the less usual representation for the Dirac matrices

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \ i = 1, 2, 3,$$

repeat the construction for the positive energy case. **[5]**

Derive the zero momentum limit of this last result. **[2]**

3. Describe Dirac hole theory and explain how it predicts the existence of the positron. **[5]**

Discuss the ideas of "bare" charge and "physical" charge. **[3]**

Describe one observable consequence of the difference between "bare" charge and "physical" charge. **[2]**

You may ***assume*** that the charge conjugation operation on a Dirac spinor is $\psi \rightarrow \psi_c = C\gamma^0\psi^*$ where $*$ denotes complex conjugation and $C = i\gamma^2\gamma^0$.

Determine the behaviour under charge conjugation of the Dirac covariant $\overline{\psi}\psi$. **[4]**

Also, determine the behaviour under charge conjugation of the Dirac covariants $\overline{\psi}\gamma^0\gamma^i\psi, \ i = 1, 2, 3$ **[6]**

(You may ***assume*** that C has the properties

$$C^\dagger = -C, C^2 = -I, \gamma^0 C\gamma^0 = -C \quad \text{and} \quad \gamma^\mu C = -C(\gamma^\mu)^T$$

Where T denotes transpose and † denotes Hermitian adjoint)

PART MARKS

4. Give an informal derivation of the differential equation obeyed by the propagator $\hat{S}_F(x';x)$ for a relativistic electron in an electromagnetic field. Show that this reduces to the equation

$$(i\nabla\!\!\!/' - m)\ S_F\ (x';x) = \delta^4(x'-x)\ \mathrm{I}$$

for the propagator $S_F\ (x';x)$ for a free electron, where the symbols have their usual meanings. **[4]**

Show that for $p^2 \neq m^2$ the corresponding momentum space propagator $\widetilde{S}_F(p)$ is given by

$$\widetilde{S}_F(p) = (p\!\!\!/ - m)^{-1}$$

where the symbols have their usual meanings. **[4]**

Assuming that the general result is $\widetilde{S}_F(p) = \dfrac{(p\!\!\!/ + m)}{(p^2 - m^2 + i\varepsilon)}$

where ε is a small positive quantity that tends to zero, show that $S_F(x';x)$ contains only positive frequency components when $t' > t$. **[12]**

5. Starting from the S-matrix element to first order in the interaction in propagator theory, show that the S-matrix element for electron scattering from a Dirac proton may be written as

$$S_{fi} = -ie^2 \int d^4x \int d^4y \overline{\psi}_f(x)\gamma_\mu \psi_i(x) D_F(x-y) \overline{\psi}_f^p(y)\gamma^\mu \psi_i^p(y)$$

where D_F is the photon propagator, ψ_i and ψ_f are the free particle wave functions for the initial and final electrons and ψ_i^p and ψ_f^p are the free particle wave functions for the initial and final protons. **[13]**

Perform the x and y integrations to obtain an explicit expression for the S-matrix element in terms of Dirac spinors with the wave functions normalised in a box of volume V. **[7]**

You may ***assume*** that the momentum space photon propagator $\widetilde{D}_F(q)$ is given by

$$\widetilde{D}_F(q) = \frac{-1}{(q^2 + i\varepsilon)}$$

where ε is a small positive quantity which tends to zero.

You may ***not*** assume the Feynman Rules.

END