Royal Holloway

UNIVERSITY OF LONDON

MSci EXAMINATION

RELATIVISTIC QUANTUM MECHANICS CP4241A

SUMMER 1999

Time Allowed: TWO AND A HALF HOURS

Answer **THREE** questions only. No credit will be given for attempting any further questions.

Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin.

TURN OVER WHEN INSTRUCTED

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	m_0	=	$4\pi \times 10^{-7}$	H m^{-1}
Permittivity of vacuum	e_0	=	8.85×10^{-12}	F m^{-1}
	$1/4\pi e_0$	=	9.0×10^{9}	m F^{-1}
Speed of light in vacuum	c	=	3.00×10^{8}	m s^{-1}
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	C kg^{-1}
Planck constant	h	=	6.63×10^{-34}	J s
	h = $h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	J K^{-1}
Stefan-Boltzmann constant	s	=	5.67×10^{-8}	W m^{-2} K^{-4}
Gas constant	R	=	8.31	J mol^{-1} K^{-1}
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	N m^{2} kg^{-2}
Acceleration due to gravity	g	=	9.81	m s^{-2}
Volume of one mol of an ideal gas at STP		=	2.24×10^{-2}	m^{3}
One standard atmosphere	P_0	=	1.01×10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

e = 2.718 π = 3.142 $\log_e 10 = 2.303$

PART MARKS

1. Use rotational invariance of the hydrogen atom Hamiltonian to show that all the orbital angular momentum states with eigenvalue $l(l+1)\hbar^2$ of $\hat{L}^2$ are degenerate in energy. **[4]**

Assume that the hydrogen atom Hamiltonian can be written in the form

$$\hat{H} = -\frac{1}{4}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \left(\hat{I}^2 + \hat{K}^2 + \frac{1}{2}\right)^{-1}$$

in units where $\hbar = 1$ and the mass of the electron is $m = 1$, where

$\underset{\sim}{\hat{I}} = \frac{1}{2}\left(\underset{\sim}{\hat{L}} + \underset{\sim}{\hat{M}}\right)$ and $\underset{\sim}{\hat{K}} = \frac{1}{2}\left(\underset{\sim}{\hat{L}} - \underset{\sim}{\hat{M}}\right)$ with $\underset{\sim}{\hat{M}}$ the Lenz vector.

Also a*ssume* that $\underset{\sim}{\hat{I}}$ and $\underset{\sim}{\hat{K}}$ commute with each other, separately obey an angular momentum algebra and that $\underset{\sim}{\hat{L}} \cdot \underset{\sim}{\hat{M}} = 0$.

Show that the energy levels of the hydrogen atom may be written in the form

$$E_n = -\frac{1}{2n^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$$

where n is an integer, and that they have degeneracy n^2. **[10]**

Show that, for a fixed value of n, the allowed values of the angular momentum quantum number l run from 0 to n-1 in integer steps. **[3]**

Show that this last result also implies degeneracy n^2. **[3]**

TURN OVER

PART MARKS

2. Discuss the helicity operator for a Dirac particle. **[3]**

Discuss in what way we need to modify the solutions of the Dirac equation to be able to describe neutrinos and anti-neutrinos. **[3]**

Assume that the Dirac matrices may be represented by

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad i = 1, 2, 3,$$

where σ^i are the 2 x 2 Pauli spin matrices.

Using this representation for the Dirac matrices, and writing $\psi = e^{-i\,p\cdot x}\begin{pmatrix} \phi \\ \chi \end{pmatrix}$, for positive energy plane wave solutions of the Dirac equation, where ϕ and χ are 2 component column matrices, derive from the Dirac equation explicit equations for ϕ and χ for the case of a neutrino. Also carry through the corresponding calculation for the case of an anti-neutrino.

[5]

Hence, show how to construct spinors to describe the neutrino and anti-neutrino. **[9]**

(You may assume that $\gamma^0 = \beta$ and $\gamma^i = \beta\,\alpha^i$ in the usual notation. You may also assume that $\sigma^1\,\sigma^2\,\sigma^3 = i\,I$.)

3. Describe Dirac hole theory and discuss the production of an electron-positron pair by a photon using it. **[7]**

Using Dirac hole theory discuss how negative energy solutions of the Dirac equation can be used to represent anti-particles. **[3]**

The charge conjugation operation on a Dirac spinor is $\psi \to \psi_c = C\,\gamma^0\,\psi^*$ where * denotes complex conjugate and $C = i\,\gamma^2\,\gamma^0$. Derive the behaviour under charge conjugation of $\overline{\psi}\,\gamma^\mu\psi$ and $\overline{\psi}\,\gamma^\mu\,\gamma_5\,\psi$ where $\gamma_5 = i\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$. **[7]**

Hence, discuss why charge conjugation invariance is broken by the weak interactions. **[3]**

(You may *assume* C has the properties
$C^\dagger = -C, \quad C^2 = -I, \quad \gamma^0\,C\,\gamma^0 = -C$ and $\gamma^\mu C = -C\left(\gamma^\mu\right)^T$ where T denotes transpose and † denotes Hermitian adjoint. You may also assume that $\gamma_5^T = \gamma_5^\dagger = \gamma_5$ and $\left\{\gamma_5, \gamma^\mu\right\} = 0$.)

PART MARKS

4. For non-relativistic quantum mechanics, give an argument that the wave functions at times t and t' are related by

$$\psi(t',\mathbf{x}') = i\int d^3x\, G(t',\mathbf{x}';t,\mathbf{x})\,\psi(t,x),$$ **[4]**

for $t' > t$, where G is the propagator.

When the interaction potential V is turned on at time t_1 for a short time interval Δt_1, show that the wave function $\psi(t',\mathbf{x}')$ at time $t' > t_1 + \Delta t_1$ is related to the free particle wave function φ by

$$\psi(t',\mathbf{x}') = i\int d^3x\, G_0(x';x)\,\phi(x) + \int d^3x_1\; G_0(x';x_1)\,V(x_1)\,\phi(x_1)\,\Delta t_1$$

where G_0 is the free propagator and $(t, \mathbf{x})$ has been abbreviated to x, on the right-hand-side. **[12]**

Deduce a relationship between the propagator and the free propagator. **[4]**

5. Starting from the S-matrix element to first order in the interaction in propagator theory, show that the S-matrix element for electron scattering from a Dirac proton may be written as

$$S_{fi} = -ie^2\int d^4x\int d^4y\,\overline{\psi}_f(x)\gamma_\mu\psi_i(x)\,D_F(x-y)\,\overline{\psi}_f^P(y)\gamma^\mu\psi_i^P(y)$$

where D_F is the photon propagator, ψ_i and ψ_f are the free particle wave functions for the initial and final electrons, and ψ_i^P and ψ_f^P are free particle wave functions for the initial and final protons. **[16]**

Derive an explicit expression for the momentum space photon propagator. **[4]**

You may **not** assume the Feynman rules.

END