UNIVERSITY OF LONDON

MSci EXAMINATION 2000

For Internal Students of

Royal Holloway

<u>DO NOT TURN OVER UNTIL TOLD TO BEGIN</u>

PH4241A: RELATIVISTIC QUANTUM MECHANICS

Time Allowed: TWO AND A HALF hours

Answer THREE QUESTIONS only

No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

Calculators ARE permitted

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	m_0	=	$4\pi \times 10^{-7}$	H m^{-1}
Permittivity of vacuum	e_0	=	8.85×10^{-12}	F m^{-1}
	$1/4\pi e_0$	=	9.0×10^{9}	m F^{-1}
Speed of light in vacuum	c	=	3.00×10^{8}	m s^{-1}
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	C kg^{-1}
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	J K^{-1}
Stefan-Boltzmann constant	s	=	5.67×10^{-8}	W m^{-2} K^{-4}
Gas constant	R	=	8.31	J mol^{-1} K^{-1}
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	N m^2 kg^{-2}
Acceleration due to gravity	g	=	9.81	m s^{-2}
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^3
One standard atmosphere	P_0	=	1.01×10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

$e = 2.718$ $\pi = 3.142$ $\log_e 10 = 2.303$

PART MARKS

1. Angular momenta $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ are combined to obtain the total angular momentum

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2 .$$

Derive the allowed values of the quantum number j for $\hat{\mathbf{J}}^2$ in terms of the quantum numbers j_1 and j_2 for $\hat{\mathbf{J}}_1^2$ and $\hat{\mathbf{J}}_2^2$, respectively, in the usual notations. **[7]**

Angular momenta $j_1 = 2$ and $j_2 = 1$ are combined.

Construct the eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_z$

$$| j = 3, \quad m = 3\rangle$$ **[3]**

$$| j = 3, \quad m = 2\rangle$$ **[5]**

and $$| j = 2, \quad m = 2\rangle$$ **[5]**

in terms of eigenstates of $\hat{\mathbf{J}}_1^2$, $\hat{\mathbf{J}}_{1z}$, $\hat{\mathbf{J}}_2^2$ and $\hat{\mathbf{J}}_{2z}$.

(You may assume that

$$\hat{\mathbf{J}}_- \,| j, m\rangle = \sqrt{(j-m+1)(j+m)}\;\hbar\,| j, m-1\rangle$$

where the symbols have their usual meanings.)

PART MARKS

2. Give an informal derivation of the Klein-Gordan equation. **[3]**

Show that the probability density for the Klein-Gordan wave function ϕ is

$$\rho = i\left(\phi^* \frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)$$

and that the probability current is

$$\mathbf{J} = -ic^2\left(\phi^*\tilde{\mathbf{N}}\phi - \phi\tilde{\mathbf{N}}\phi^*\right).$$ **[8]**

Check that the Klein-Gordan equation has plane wave solutions of the form

$$Ne^{\pm i\, p.x}$$

where N is a normalisation factor. **[4]**

Derive the probability densities for these plane wave solutions. **[3]**

Discuss why the Dirac interpretation of negative energy solutions does not work for the Klein-Gordan equation and state the Feynman interpretation. **[2]**

3. Describe what is meant by a symmetry of the Dirac equation. **[2]**

The charge conjugation symmetry operation acting on a Dirac spinor is

$$\psi \to \psi_c = C\,\gamma^0 \psi^*$$

where * denotes complex conjugation and

$$C = i\,\gamma^2\,\gamma^0 .$$

Prove that

$$C\,\gamma^0\, u^*(\mathbf{p},1) = v(\mathbf{p},1)$$ **[8]**

where the Dirac spinor $u(\mathbf{p},1)$ is defined by

$$u(\mathbf{p},1) = (E+m)^{1/2}\begin{pmatrix} \phi^1 \\ \dfrac{\boldsymbol{\sigma}.\mathbf{p}}{E+m}\,\phi^1 \end{pmatrix}$$

with

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the Dirac spinor $v(\mathbf{p},1)$ is defined by

$$v(\mathbf{p},1) = (E+m)^{1/2}\begin{pmatrix} \dfrac{\boldsymbol{\sigma}.\mathbf{p}}{E+m}\,\chi^1 \\ \chi^1 \end{pmatrix}$$

with

$$\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Discuss why this was the result to be expected for the charge conjugation operation. **[2]**

Derive the behaviour under charge conjugation of the Dirac covariant $\bar{\psi}\,\psi$. **[8]**

(You may assume that $C^\dagger = -C$).

4. For an electron scattering from the fixed Coulomb potential of a nucleus of atomic number Z the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{|\mathbf{q}|^4} \left| \bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i) \right|^2$$

evaluated at $E_f = E_i$ and $|\mathbf{p}_\mathrm{f}| = |\mathbf{p}_\mathrm{i}|$, where the symbols have their usual meanings.

If the final spin state is not observed and the incident beam of protons is unpolarised, show that the differential cross section may be rewritten in terms of a matrix trace as

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{2|\mathbf{q}|^4} Tr\left((\not{p}_f + m)\gamma^0 (\not{p}_i + m)\gamma^0\right).$$ **[12]**

By using trace theorems, show that $\frac{d\sigma}{d\Omega}$ may be simplified to

$$\frac{d\sigma}{d\Omega} = \frac{2Z^2\alpha^2}{|\mathbf{q}|^4} \left[m^2 + 2E_f E_i - p_f . p_i\right].$$ **[8]**

PART MARKS

5. The neutral Klein-Gordan field has Lagrangian density L given by

$$L = \frac{1}{2}\left(\frac{\partial\phi}{\partial x^{\mu}} \frac{\partial\phi}{\partial x_{\mu}} - m^2\phi^2 \right).$$

The field may be expanded in the form

$$\phi(x) = \int d^3k \left[a(k)\, f_k^{(+)}(x) + a^{\dagger}(k)\, f_k^{(-)}(x) \right]$$

where

$$f_k^{(\pm)}(x) = \frac{1}{\sqrt{(2\pi)^3 (2E_k)}} e^{\mp ik.x}.$$

Obtain the Hamiltonian density in terms of ϕ and its derivatives. **[3]**

Show that the Hamiltonian can be written in the form

$$H = \frac{1}{2}\int d^3k\, E_k \left(a(k)\, a^{\dagger}(k) + a^{\dagger}(k)\, a(k) \right).$$ **[13]**

Also show that

$$[H, a(k)] = -E_k\, a(k).$$ **[4]**

(You may assume that

$$[a(k), a(k')] = 0$$

and that $$\left[a(k), a^{\dagger}(k') \right] = \delta(\mathbf{k} - \mathbf{k}').)$$

END