Royal Holloway

UNIVERSITY OF LONDON

MSci EXAMINATION

RELATIVISTIC QUANTUM MECHANICS AND
ANGULAR MOMENTUM CP4240A

SUMMER 1998

Time Allowed: TWO HOURS

Answer **TWO** questions only. No credit will be given for attempting a further question.

Each question carries 20 marks. The mark *provisionally allocated* to each section is indicated in the margin.

TURN OVER WHEN INSTRUCTED

PART MARKS

1. Two angular momenta $\hat{\underset{\sim}{J}}_1$ and $\hat{\underset{\sim}{J}}_2$ are combined to total angular momentum

$$\hat{\underset{\sim}{J}} = \hat{\underset{\sim}{J}}_1 + \hat{\underset{\sim}{J}}_2.$$

Derive the allowed quantum numbers j and m of $\hat{\underset{\sim}{J}}$ in terms of the quantum numbers j_1, m_1 of $\hat{\underset{\sim}{J}}_1$ and j_2, m_2 of $\hat{\underset{\sim}{J}}_2$, in the usual notations. **[9]**

If angular momenta with $j_1 = 1$ and $j_2 = 1$ are combined, construct the eigenstates of the total angular momentum $\hat{\underset{\sim}{J}}$,

$$|\, j = 2, m = 2\rangle, \quad |\, j = 2, m = 1\rangle \quad \text{and} \quad |\, j = 2, m = 0\rangle,$$

in terms of eigenstates of $\hat{\underset{\sim}{J}}{}_1^2$, $\hat{J}_{1z}$, $\hat{\underset{\sim}{J}}{}_2^2$ and $\hat{J}_{2z}$. **[11]**

(You may assume that

$$\hat{J}_-|\, j, m\rangle = \sqrt{(j-m+1)(j+m)}\;\hbar|\, j, m-1\rangle$$

where the symbols have their usual meanings.)

2. Using the representation for the Dirac matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

construct the general positive energy plane wave solution of the free particle Dirac equation. **[12]**

Repeat the construction using the less usual representation for the Dirac matrices

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad i = 1, 2, 3,$$ **[6]**

Also, in each case, derive the non-relativistic limit of your solution. **[2]**

PART MARKS

3. Starting from the Dirac equation for a free particle, derive the Dirac equation with a four-vector electromagnetic potential A^{μ} present. Show that it can be written in the form

$$i\frac{\partial\psi}{\partial t} = \left(\underset{\sim}{\alpha}.\underset{\sim}{\hat{\pi}} + \beta m\right)\psi - qA^{0}\psi$$

where $\underset{\sim}{\hat{\pi}} = -i\underset{\sim}{\nabla} + q\underset{\sim}{A}$ and the symbols have their usual meanings. **[9]**

Assume that in the non-relativistic limit we can write ψ in terms of two component spinors ϕ and χ as

$$\psi = e^{-imt}\begin{pmatrix}\phi \\ \chi\end{pmatrix}$$

where ϕ and χ vary slowly with time. Assuming also that $m >> qA^{0}$, show that

for $\beta = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$ and $\underset{\sim}{\alpha} = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma} & 0\end{pmatrix}$

ϕ obeys

$$i\frac{\partial\phi}{\partial t} = -qA^{0}\phi + \frac{\left(\underset{\sim}{\sigma}.\underset{\sim}{\hat{\pi}}\right)^{2}}{2m}\phi.$$ **[6]**

Given that this last equation may be simplified to

$$i\frac{\partial\phi}{\partial t} = \left(-qA^{0} + \frac{\left(\underset{\sim}{\hat{p}} + q\underset{\sim}{A}\right)^{2}}{2m} + \frac{q\underset{\sim}{\sigma}.\underset{\sim}{B}}{2m}\right)\phi$$

with $\underset{\sim}{B}$ the magnetic field, derive an expression for the spin magnetic moment of a Dirac particle. **[5]**

END