UNIVERSITY OF LONDON

(University College London)

M.Sci. DEGREE 1998

PHYSICS CP4230: Basic Scattering Theory

N.B. Examination time = 2 hours

Answer **TWO** *questions.*

Numbers in square brackets indicate the provisional allocation of marks per question section.

1. The Green's function for the Helmholtz equation $(\nabla^2+k^2)G(\vec{r},\vec{r}\,')=\delta^3(\vec{r}-\vec{r}\,')$ has two solutions

$$G^{\pm}(\vec{r},\vec{r}\,')=-\frac{1}{4\pi}\frac{\exp(\pm ik|\vec{r}-\vec{r}\,'|)}{\vec{r}-\vec{r}\,'}\,.$$

Use this to show that the Schrödinger equation for the scattering of a particle of mass m and wave vector $\vec{k}$ by a potential $V(\vec{r})$ is equivalent to the integral equation

$$\psi_{\vec{k}}(\vec{r})=\exp(i\vec{k}\cdot\vec{r})+\frac{2m}{\hbar^2}\int G^{+}(\vec{r},\vec{r}\,')\,V(\vec{r}\,')\,\psi_{\vec{k}}(\vec{r}\,')\,d^3r'$$

for the wave function $\psi_{\vec{k}}(\vec{r})$ with the correct asymptotic behaviour. **[7 marks]**

Hence deduce that the scattering amplitude may be written as

$$f(\vec{k}',\vec{k})=-\frac{m}{2\pi\hbar^2}\int\exp(-i\vec{k}'\cdot\vec{r})V(\vec{r})\,\psi_{\vec{k}}(\vec{r})\,d^3r\,.$$ **[3 marks]**

Further show that the first Born approximation for the scattering with momentum transfer $\hbar\vec{q}=\hbar(\vec{k}-\vec{k}')$ from a spherically symmetric potential is given by

$$f^B(q)=-\frac{2m}{\hbar^2}\int_0^\infty r^2\,dr\,\frac{\sin(qr)}{qr}\,V(r)\,.$$ **[2 marks]**

Evaluate $f^B(q)$ for the square well potential where $V(r)=V_0$ for $r\le R$ and which vanishes for $r>R$. **[4 marks]**

By taking the limit of $q\to 0$ in the integrand, or otherwise, derive the value of the first Born approximation for the square well at $q=0$. Hence show how the optical theorem can be used to obtain the imaginary part of the second Born scattering amplitude at <u>low</u> energies. **[4 marks]**

2. A particle of wave number k is scattered by an attractive delta–shell potential situated at $r = R$. Although the reduced s–wave function $u(r)$ is continuous at R, the first derivative exhibits a jump

$$\left.\frac{du(r)}{dr}\right|_{R+\epsilon} - \left.\frac{du(r)}{dr}\right|_{R-\epsilon} = -\lambda u(R)\,.$$

By taking the interior and exterior wave functions to be of the form $\sin(kr)$ and $\sin(kr + \delta_0)$ respectively, show that the scattering phase shift satisfies

$$k \cot \delta_0 = -k \cot(kR) + \frac{k^2}{\lambda \sin^2(kR)}\,\cdot$$

[7 marks]

By letting $k \to 0$, show that the coefficients a and r_0 in the effective range expansion

$$k \cot \delta_0 = -\frac{1}{a} + \tfrac{1}{2} r_0 k^2 + 0(k^4)$$

are given by

$$a = \frac{\lambda R^2}{\lambda R - 1} \quad \text{and} \quad r_0 = \frac{2(1 + \lambda R)}{3\lambda}\,\cdot$$

[5 marks]

Keeping only the first two terms in the effective range expansion, construct explicitly the s–wave scattering amplitude

$$f_0 = \frac{1}{k} e^{i\delta_0} \sin \delta_0 = \frac{1}{k \cot \delta_0 - ik}$$

as a function of λ and R. **[2 marks]**

A pole of f_0 for $k = i\gamma$, where $\gamma > 0$, corresponds to an s–wave bound state of the system. Solve for γ and discuss the range of values of λ and R for which one can reliably predict the existence of a bound state from the approximate formula. **[6 marks]**

You may need

$\sin(A + B) = \sin A \cos B + \cos A \sin B$ and $\cos(A + B) = \cos A \cos B - \sin A \sin B$

and the series expansions for small x

$$\sin x \approx x - \frac{x^3}{6} \quad \text{and} \quad \cos x \approx 1 - \frac{x^2}{2}\,\cdot$$

3. In first-order time-dependent perturbation theory, the probability amplitude that a system in an initial state s is to be found in another state n at time t through the action of a perturbation H' is given by

$$c_n(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} H'_{ns}\, e^{i\omega_{ns}t'}\, dt' \,,$$

where H'_{ns} is the matrix element of the perturbation and $\omega_{ns} = (E_n - E_s)/\hbar$ is the frequency difference between states of energy E_n and E_s.

Use this result, together with a 'slow switching-on' procedure, to prove Fermi's Golden Rule

$$,_{ns} = \frac{2\pi}{\hbar}\,\delta(E_s - E_n)\,\left|H'_{ns}\right|^2$$

for the transition rate for a time-independent perturbation. **[8 marks]**

A particle of mass m, initially in the ground state of a one-dimensional harmonic oscillator potential, is perturbed by a travelling pulse

$$H' = \lambda\,\delta(x - vt)\,,$$

where λ and v are constants, with v being the speed of the pulse. Use first-order perturbation theory to show that the probability P_1 that the particle will end up at large positive time in the first excited state is

$$P_1 = \frac{\lambda^2\alpha^2}{2m^2v^4}\,\exp(-\hbar\omega/2mv^2)\,.$$

[8 marks]

Find the speed v for which P_1 is a maximum. Give a qualitative argument why this speed gives the largest effect. **[4 marks]**

The ground and first excited state wave functions of the harmonic oscillator are

$$\Phi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\exp(-\tfrac{1}{2}\alpha^2x^2)\,, \quad \Phi_1(x) = \sqrt{2}\,\alpha\, x\,\Phi_0(x)\,,$$

where $\alpha = \sqrt{m\omega/\hbar}$.

Integrals of the form $\int_{-\infty}^{\infty} t^n\, e^{-\beta^2t^2}\, e^{i\gamma t} dt$ can be most easily evaluated by completing the square in the exponent.

END OF PAPER