UNIVERSITY OF LONDON
(University College London)

M.Sci. DEGREE 1997

PHYSICS CP4230: Basic Scattering Theory

N.B. Examination time = 2 hours

Answer **TWO** *questions.*

Numbers in square brackets indicate the provisional allocation of marks per question section.

1. Show that

$$G^{(\pm)}(x,x') = -\frac{i}{2k}\,\exp(\pm ik|x-x'|)$$

satisfies the one-dimensional Green's function equation

$$\frac{d^2G^{(\pm)}(x,x')}{dx^2} + k^2\,G^{(\pm)}(x,x') = \delta(x-x')\,.$$ **[5 marks]**

[Hint: Show that away from $x = x'$ it satisfies the equation without the delta-function. By integrating the equation from $x'-\epsilon$ to $x'+\epsilon$, where ϵ is arbitrarily small but positive, show that it is still satisfied by the given form.]

The wave function $\psi(x)$, describing a beam of particles of mass m and wave number k incident from the left, satisfies the boundary conditions

$$\psi(x) = \begin{array}{ll} e^{ikx} + \;R\,e^{-ikx} & \text{as } x\to-\infty\,, \\ \qquad\quad T\,e^{+ikx} & \text{as } x\to+\infty\,. \end{array}$$

Show that the one-dimensional Schrödinger equation can be converted into an integral equation with these boundary conditions incorporated to give

$$\psi(x) = e^{ikx} - \frac{i}{2k}\int_{-\infty}^{+\infty} e^{+ik|x-x'|}\,U(x')\,\psi(x')\,dx'\,,$$

where $U(x)$ is the reduced potential, which is assumed to vanish at large values of $|x|$. **[5 marks]**

Derive integral expressions for the transmission and reflection amplitudes T and R analogous to that for the three-dimensional scattering amplitude f. Hence obtain expressions for the first Born approximation for T and R. **[3 marks]**

Solve the integral equation for the δ-function potential

$$U(x) = \lambda\,\delta(x)\,,$$

and hence obtain expressions for the transmission and reflection amplitudes. **[4 marks]**

Compare these expressions with those obtained to first order in λ, *i.e* the first Born approximation. **[3 marks]**

2. The S-wave Schrödinger equation for the scattering of a particle with wave number k from a reduced potential $U(r)$ of finite range is

$$\left\{ \frac{d^2}{dr^2} + k^2 - U(r) \right\} u(k,r) = 0 \, .$$

Choose the radial wave function which vanishes at the origin to have the normalisation

$$u(k,r) \to \frac{\sin(kr+\delta)}{\sin\delta}$$

as $r \to \infty$, where δ is the S-wave phase shift, and adopt a form for the free solution $v(k,r)$ with a similar asymptotic behaviour. Hence derive the relation

$$k_2 \cot\delta_2 - k_1 \cot\delta_1 = (k_2^2 - k_1^2) \int_0^\infty (v_1 v_2 - u_1 u_2)\, dr \, , \qquad \textbf{[6 marks]}$$

where the subscripts 1 and 2 refer to solutions at wave numbers k_1 and k_2 respectively.

By letting $k_2 \to 0$, obtain the effective range expansion

$$k \cot\delta = -\frac{1}{a} + \tfrac{1}{2} r_0 k^2 + O(k^4) \, ,$$

and derive an integral expression for the effective range r_0 in terms of the full and free wave functions evaluated at zero energy. **[4 marks]**

The S-wave function for the Yamaguchi potential is given by

$$u(r) = \frac{\sin(kr+\delta)}{\sin\delta} - e^{-\beta r} \, ,$$

where the phase shift is determined by

$$k \cot\delta = \frac{4\beta^2 k^2 - (\beta^2 - k^2)(\alpha^2 + 2\alpha\beta + k^2)}{2\beta(\alpha+\beta)^2} \, ,$$

with α and β constants.

By comparing with the effective range expansion, determine the values of a and r_0. **[4 marks]**

Compare this value of r_0 with the result obtained by using the integral representation that you have previously derived. **[6 marks]**

3. A particle of mass m is initially bound in the state i of an unperturbed Hamiltonian H_0. Show that, in lowest order time-dependent perturbation theory, the transition amplitude $c_f(t)$ for the excitation of this to a bound final state f by a time-dependent perturbing Hamiltonian H' is given by

$$c_f(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} H'_{fi}\, \exp\left(\frac{i(E_f - E_i)t'}{\hbar}\right)\, dt' \qquad (f \neq i)\,,$$

where H'_{fi} is the matrix element of the perturbation and E_f and E_i are the energies of the two states. **[10 marks]**

A hydrogen atom is subjected to a time-dependent perturbation

$$H'(\vec{r}, t) = \frac{Az}{\pi(\lambda^2 + t^2)}\,,$$

where A and λ are real constants, with $\lambda > 0$, and $\vec{r}$ is the position of the electron with third component z. If at time $t = -\infty$ the atom is in the (1s) ground state, find in first-order perturbation theory the probability, P_1, that it will be in the 2s state after a long time $t \to +\infty$. **[2 marks]**

Show that the probability, P_2, that it is eventually in the 2p state is given by

$$P_2 = \frac{2^{15}}{3^{10}}\left(\frac{a_0 A}{\hbar\lambda}\right)^2 \exp(-2\omega\lambda)\,,$$

where

$$\omega = \frac{3}{8}\frac{e^2}{4\pi\epsilon_0 a_0 \hbar}\,,$$

and a_0 is the Bohr radius. **[8 marks]**

Take the proton to be infinitely heavy and neglect any effects due to the electron spin.

The hydrogenic wave functions $\psi_{n\ell m}(r,\theta,\phi) = R_{nl}(r)\, Y_\ell^m(\theta,\phi)$ may be obtained for the relevant states from

$$R_{10}(r) \;=\; 2\left(\frac{1}{a_0}\right)^{3/2} \exp(-r/a_0)\,,$$

$$R_{20}(r) = 2\left(\frac{1}{2a_0}\right)^{3/2}\left(1-\frac{r}{2a_0}\right)\exp(-r/2a_0)\,,$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2}\frac{r}{a_0}\exp(-r/2a_0)\,,$$

$$Y_0^0(\theta,\phi) = \left(\frac{1}{4\pi}\right)^{1/2}\,,\quad Y_1^0(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta,\quad Y_1^{\pm 1}(\theta,\phi) = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\phi},$$

and the corresponding energy levels are

$$E_n = -\frac{e^2}{4\pi\epsilon_0 a_0}\frac{1}{n^2}\,.$$

Note that for any parameter $b > 0$,

$$\int_0^\infty x^n \exp(-bx) = \frac{n!}{b^{n+1}}\,,\quad \text{and}\quad \int_{-\infty}^{+\infty}\frac{\exp(i\omega t)}{b^2+t^2}\,dt = \frac{\pi}{b}\exp(-\omega b)\,.$$

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