Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

[Part marks]

1. Show that, for any normalizable function ϕ that satisfies the appropriate boundary conditions, the expectation value of the Hamiltonian operator H describing a one-dimensional time-independent quantum mechanical system satisfies

$$\frac{\int \phi^*(x)\, H\phi(x)\, \mathrm{d}x}{\int \phi^*(x)\, \phi(x)\, \mathrm{d}x} \geq E_0,$$

where E_0 is the lowest energy eigenvalue. **[4]**

How may this expression be employed to obtain a useful limit on the ground state energy of the system?
Show that under certain circumstances, to be stated, the method can be extended to estimate the energy of the first excited state. **[4]**

A particle of mass m and charge q moving in a one-dimensional harmonic oscillator potential of angular frequency ω and subjected to a uniform electric field $\mathcal{E}$ along the x-axis is described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}x.$$

For a trial function of the form

$$\phi(x) = \mathrm{e}^{-\alpha(x-\beta)^2},$$

where α and β are **two** parameters that can be varied, show that it gives rise to an estimate of the ground state energy as

$$E = \frac{1}{2}\hbar\omega - \frac{q^2\mathcal{E}^2}{2m\omega^2}.$$ **[8]**

Show that this answer is also the **exact** ground state energy. Hence suggest, with reasons, a trial function to use for an estimate of the energy of the first excited state. **[4]**

$$\int_{-\infty}^{\infty} x^{2n}\mathrm{e}^{-\lambda x^2}\,\mathrm{d}x = \frac{(2n)!}{2^{2n}n!\lambda^n}\sqrt{\frac{\pi}{\lambda}} \quad \text{for } n = 0, 1, 2, \ldots, \text{ and } \lambda > 0, \text{ and } 0! = 1.$$

2. A system subjected to a time-dependent perturbation is described by a Hamiltonian

$$H(\mathbf{r},t) = H_0(\mathbf{r}) + \lambda H'(\mathbf{r},t),$$

where λ is a small parameter. The eigenfunctions $\phi_n(\mathbf{r})$ and energy eigenvalues E_n of $H_0(\mathbf{r})$ are known. Given that a solution of the time-dependent Schrödinger equation can be written as

$$\psi(\mathbf{r},t) = \sum_n c_n(t)\,\phi_n(\mathbf{r})\,\mathrm{e}^{-iE_nt/\hbar},$$

obtain a differential equation for the transition coefficients $c_n(t)$. **[4]**

Initially at time t_0, the system is in a definite eigenstate $\phi_i(\mathbf{r})$ with energy E_i. Show that at a later time t, to lowest order in λ, the transition amplitude $c_k(t)$ for excitation of a state $\phi_k(\mathbf{r})$ of energy E_k $(\neq E_i)$ is given by

$$c_k(t) = \frac{1}{i\hbar}\int_{t_0}^{t}\langle\phi_k(\mathbf{r})|\lambda H'(\mathbf{r},t')|\phi_i(\mathbf{r})\rangle\,\mathrm{e}^{i\omega_{ki}t'}\,\mathrm{d}t',$$

where $\omega_{ki} = (E_k - E_i)/\hbar$. **[5]**

A particle for $t \leq 0$ is in the ground state $|0\rangle$ of a one-dimensional harmonic oscillator potential of angular frequency ω. At $t = 0$ a perturbation

$$\lambda H'(x,t) = Ax^3 \sin\Omega t$$

is turned on, where A and Ω are constants. Find $c_k(t)$ in this case in terms of the matrix elements $\langle k|x^3|0\rangle$. **[3]**

Show that in first order the system can only make a transition to the first or third excited state. **[4]**

Show that for Ω near a certain frequency, the probability of finding the system in an excited state is large. Evaluate the leading term in this probability for **either** of the excited states. **[4]**

{Note the matrix elements of x for the harmonic oscillator are
$\langle k|x|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left[\sqrt{n}\delta_{k,n-1} + \sqrt{n+1}\delta_{k,n+1}\right]$. }

3. If $\mathbf{J}_1$ and $\mathbf{J}_2$ are two angular momenta the operators $\mathbf{J}_1^2$, $\mathbf{J}_2^2$, J_{1z} and J_{2z} form a commuting set with eigenvalues $j_1(j_1+1)\hbar^2$, $j_2(j_2+1)\hbar^2$, $m_1\hbar$ and $m_2\hbar$ respectively and eigenvectors $|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle|j_2, m_2\rangle$. If $\mathbf{J}$ is their sum, $(\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2)$, show that $\mathbf{J}_1^2$, $\mathbf{J}_2^2$, $\mathbf{J}^2$ and J_z also form a commuting set . **[3]**

{You may assume the commutation relations $[J_{nx}, J_{ny}] = i\hbar J_{nz}$, and $[\mathbf{J}_n^2, J_{nk}] = 0$ for $n = 1, 2$ and $k \equiv x, y, z$.}

Their common eigenstates $|j_1, j_2, J, M\rangle$ are related to $|j_1, j_2, m_1, m_2\rangle$ by a unitary transform. Show that the maximum and minimum values of J are

$$J_{\max} = j_1 + j_2 \qquad \text{and} \qquad J_{\min} = |j_1 - j_2|.$$ **[4]**

{Note that $\sum_{r=n}^{N} r = \frac{1}{2}(N^2 + N + n - n^2)$. }

By using the angular momentum ladder operators $J_\pm = J_x \pm iJ_y$, show that the eigenstates of the total angular momentum of two spin-$\frac{1}{2}$ particles are given by

$$\begin{aligned} |1,1\rangle &= \alpha_1\alpha_2 \\ |1,0\rangle &= \sqrt{\tfrac{1}{2}}(\alpha_1\beta_2 + \beta_1\alpha_2) \\ |1,-1\rangle &= \beta_1\beta_2 \end{aligned} \qquad \text{and} \qquad |0,0\rangle = \sqrt{\tfrac{1}{2}}(\alpha_1\beta_2 - \beta_1\alpha_2),$$

where $\alpha_{i,}$, β_i are the eigenstates of J_{iz} $(i = 1, 2)$ with eigenvalues $\hbar/2$ and $-\hbar/2$ respectively. **[4]**

{You may assume the general result $J_\pm|j,m\rangle = [j(j+1) - m(m\pm1)]^{1/2}\hbar|j, m\pm1\rangle$. }

Consider the collision between two spin-$\frac{1}{2}$ particles, where the spin interaction is treated quantum mechanically and other features are treated classically. The interaction between the spins $\mathbf{s}_1$ and $\mathbf{s}_2$ is of the form

$$H'(t) = a(t)\,\mathbf{s}_1\cdot\mathbf{s}_2,$$

where $a(t)$ depends on the separation of the two particles. Before the collision the initial state of the two-particle system is $|\psi(t = -\infty)\rangle = \alpha_1\beta_2$. If, for simplicity, $a(t)$ is approximated by

$$a(t) = \begin{cases} 0 & t < 0 \\ a_0 & 0 \le t \le T, \\ 0 & > T \end{cases}$$

where T is the effective collision time and a_0 is a constant, show that after the collision the spin state is

$$|\psi(t = +\infty)\rangle = e^{i\Omega T/4}\{\alpha_1\beta_2\cos(\Omega T/2) - i\beta_1\alpha_2\sin(\Omega T/2)\},$$

with $\Omega = a_0\hbar$. **[6]**

If observer 1 measures s_{1z} and gets the value $-\hbar/2$, what is the probability of this occurring? If observer 2 at a later time measures s_{2z} what are the probabilities that the values $\hbar/2$ or $-\hbar/2$ are obtained? **[3]**

4. The rate of transition, W_{ki}, of a system from an initial state $|\psi_i(\mathbf{r})\rangle$ of energy E_i to a group of states, $|\psi_k(\mathbf{r})\rangle$ of mean energy E_k and energy density of states $\rho(E_k)$ is, to lowest order in the perturbation H' causing the transition, given by (Fermi's Golden Rule)

$$W_{ki} = \frac{2\pi}{\hbar} |\langle\psi_k(\mathbf{r})|H'|\psi_i(\mathbf{r})\rangle|^2 \rho(E_k) \qquad \text{with} \qquad E_k = E_i.$$

Discuss how this result may be used to obtain the Born approximation for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int V(\mathbf{r})\, \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}\, \mathrm{d}^3\mathbf{r}\right|^2,$$

describing the scattering of a monoenergetic beam of spinless particles of mass m from a weak potential $V(\mathbf{r})$, where $\hbar\mathbf{q}$ is the momentum transfer. **[8]**

Obtain an expression for the differential cross section if

$$V(\mathbf{r}) = \begin{cases} -V_0, & 0 \le r \le a; \quad V_0 > 0 \\ 0, & r > a. \end{cases}$$

Sketch the behaviour of $\mathrm{d}\sigma/\mathrm{d}\Omega$ as a function of $x = qa$. **[6]**

How may the result for the differential cross section be used to measure a?

If the particle were a neutron ($m = 940$ MeV/c^2) scattering from a nucleus of radius 5 fm, approximately how high must the incident energy be for the differential cross section to be used to measure a, assuming that the Born approximation is valid at this energy? **[4]**

Show that at high energies the scattering is concentrated to the forward direction. **[2]**

{Note for small x, $\sin x \simeq x - x^3/6$, and $\cos x \simeq 1 - x^2/2$.
The first two solutions of the transcendental equation $\tan x = x$ occur for $x = 1.430\pi$, 2.459π.
Take $\hbar c = 197$ MeV fm. }

5. The wave function

$$\psi(\mathbf{r}) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta,\phi)\frac{\mathrm{e}^{ikr}}{r}$$

is a solution for large r of the Schrödinger equation describing the elastic scattering of a monoenergetic beam of spinless particles from a short range potential $V(\mathbf{r})$. Explain briefly the physical meaning of the terms on the right-hand side. **[2]**

Evaluate the incident and scattered fluxes and show that the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega = |f(\theta,\phi)|^2$. **[5]**

{You may assume the general result that the probability current density $\mathbf{j}$ associated with a wave function $\phi(\mathbf{r})$ is given by $\mathbf{j} = \mathrm{Re}\left\{\phi^*\frac{\hbar}{mi}\boldsymbol{\nabla}\phi\right\}$, also $\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$.}

A particle of wave number k is elastically scattered by a reduced, spherical delta-shell potential $U(r) = \frac{\lambda}{a}\delta(r-a)$ at $r=a$ with $\lambda > 0$. The reduced s-wave function $u_0(r)$ satisfies the radial equation

$$\frac{\mathrm{d}^2u_0(r)}{\mathrm{d}r^2} + \left(k^2 - U(r)\right)u_0(r) = 0,$$

and although $u_0(r)$ is continuous at $r=a$, its first derivative $u_0'(r)$ is discontinuous with

$$u_0'(a+\epsilon) - u_0'(a-\epsilon) = \frac{\lambda}{a}u_0(a) \qquad \text{as } \epsilon \to 0.$$

Solve the radial equation for the potential free interior $(r<a)$ and exterior $(r>a)$ regions. Take the interior solution to have amplitude B and the exterior one to have unit amplitude.
Show that the s-wave phase shift δ_0 satisfies

$$\cot(ka+\delta_0) - \cot(ka) = \frac{\lambda}{ka}$$

and hence that

$$k\cot\delta_0 = -k\cot(ka) - \frac{k^2a}{\lambda}\mathrm{cosec}^2(ka).$$ **[6]**

By letting $k\to 0$, prove that the scattering length A_0 and effective range parameter r_0 of the effective range expansion are given by

$$A_0 = a\left(\frac{\lambda}{\lambda+1}\right) \qquad \text{and} \qquad r_0 = \frac{2a}{3}\left(\frac{\lambda-1}{\lambda}\right).$$ **[4]**

If the incident energy were such that $\tan(ka) = 0$, evaluate the phase shift δ_0, amplitude B, and the total s-wave cross section. Interpret the results physically. **[3]**

{Note: $1+\cot^2 x = \mathrm{cosec}^2 x$, $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}$, .
For small x, $\cot x \simeq \frac{1}{x} - \frac{x}{3}$, $\mathrm{cosec}\, x \simeq \frac{1}{x} + \frac{x}{6}$. }

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