Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

[Part marks]

1. A system, which at time $t = -\infty$ is in a discrete eigenstate $|i\rangle$ with energy E_i of a time-independent Hamiltonian H_0, is subjected to a weak, time-dependent perturbation $\lambda H'(t)$. Show that to lowest-order in λ the probability, $P_k(t)$, that the system will be in an eigenstate $|k\rangle$ $(k \neq i)$ of H_0 with energy E_k at a later time t is given by

$$P_k(t) = \frac{1}{\hbar^2} \left| \int_{-\infty}^{t} \langle k|\lambda H'(t')|i\rangle e^{i\omega_{ki}t'} dt' \right|^2 ,$$

where $\omega_{ki} = (E_k - E_i)/\hbar$. **[9]**

Two static spin-$\frac{1}{2}$ particles interact through a spin-spin interaction with a Hamiltonian

$$H_0 = A\,\mathbf{s}_1 \cdot \mathbf{s}_2 ,$$

where A is a positive constant and $\mathbf{s}_1$, $\mathbf{s}_2$ are the spin angular momentum operators for particles 1 and 2 respectively. The spin functions for the triplet and singlet states in terms of the eigenstates α_i, β_i with eigenvalues $\frac{1}{2}\hbar$, $-\frac{1}{2}\hbar$ respectively of the z-component of spin s_{iz} of the i-th particle, $(i = 1, 2)$, are

Triplet	Singlet
$\|1,1\rangle = \alpha_1\alpha_2$	
$\|1,0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \beta_1\alpha_2)$	$\|0,0\rangle = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$
$\|1,-1\rangle = \beta_1\beta_2$	

Show that these states are eigenstates of H_0 and deduce the energy eigenvalues. **[4]**

The system is subjected to a weak time-dependent perturbation

$$H'(t) = B(s_{1x} - s_{2x})e^{-\gamma|t|}$$

where B and γ are positive constants and s_{1x}, s_{2x} are the x-component of the spin operators. To which state or states can the system be excited, to lowest-order in B, if initially at $t = -\infty$ the system is in its lowest energy state? **[3]**

Calculate the probability that as $t \to \infty$ the system has made a transition to these states. **[4]**

The angular momentum raising and lowering operators $S_\pm = S_x \pm iS_y$ acting on a state $|s, m\rangle$ give $S_\pm|s, m\rangle = [s(s+1) - m(m \pm 1)]^{1/2}\,\hbar|s, m \pm 1\rangle$.

2. The time-independent Schrödinger equation for the motion in one-dimension of a particle of mass m and energy E in a potential $V(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x),$$

where $\psi(x)$ is the wave function. Show that, if $\psi(x)=e^{iu(x)}$, then $u(x)$ satisfies

$$\left(\frac{du}{dx}\right)^2=k^2(x)+i\frac{d^2u}{dx^2} \qquad \text{with} \qquad k^2(x)=\frac{2m}{\hbar^2}[E-V(x)] \quad \text{for} \quad E>V(x).$$ [3]

If V is a constant, a solution is $u(x)=kx$. If $V(x)$ varies slowly with x, show that the first approximation u_0 for u is

$$u_0=\pm\int^x k(x)\,dx+C_0$$

and that the next iteration $u_1(x)$ is given by

$$u_1(x)=\pm\int^x\sqrt{k^2(x)\pm ik'(x)}\,dx+C_1.$$

If $|k'(x)|\ll|k^2(x)|$, show that the WKB approximation for the wave function when $E>V(x)$ is

$$\psi(x)\approx\frac{1}{\sqrt{k(x)}}\exp\left[\pm i\int^x k(x')\,dx'\right].$$ [5]

Explain why the WKB approximation is not valid in the vicinity of a classical turning point of the motion. Outline a method whereby the WKB solution in the classically allowed region, $(E>V(x))$, may be linked to that in the classically inaccessible region, $(E<V(x))$. Do **not** give extensive mathematical details; do **NOT** derive the WKB connection formulae. [4]

Show that the WKB estimates of the energy levels of a particle of mass m in the potential

$$V(x)=-V_0+g|x|, \qquad \text{with} \qquad V_0>0 \quad \text{and} \quad g>0$$

are given by

$$E_n=-V_0+\left[\frac{9}{32}\pi^2\left(n+\tfrac{1}{2}\right)^2\frac{g^2\hbar^2}{m}\right]^{1/3}, \qquad n=0,1,2\ldots.$$ [4]

You may assume the result that in the WKB approach, approximate values of the energies of bound states can be obtained from the condition

$$\int_a^b k(x)\,dx=\left(n+\tfrac{1}{2}\right)\pi, \quad n=0,1,2\ldots,$$

where the turning points a and b are defined by $E=V(a)=V(b)$.

This potential may be used in a very crude non-relativistic model for the S-states of charmonium (a bound state of a charmed quark and an anti-charmed quark) treating the two-body problem as an effective one-body problem with a reduced mass of half the charmed quark mass, m_c. The two lowest S-states have energies of 3.1 GeV and 3.7 GeV. Obtain an estimate of the confining-potential strength parameter g in units of $\mathrm{GeV.fm^{-1}}$. **[4]**

[Take $m_c = 1.5\,\mathrm{GeV/c^2}$ and $\hbar c = 0.2\,\mathrm{GeV.fm}$.]

3. Fermi's Golden Rule gives the rate of transition, W_{Gi}, of a system from a state $|i\rangle$ of energy E_i to a narrow group of final states G of mean energy E_k and density of states $\rho(E_k)$ under the action of a perturbation H' as

$$W_{Gi} = \frac{2\pi}{\hbar} \left|\langle k\,|H'|\,i\rangle\right|^2 \rho(E_k) \qquad \text{with} \qquad E_k = E_i.$$

Discuss how this result may be applied to the elastic scattering of a beam of spinless particles of mass m by a weak, short-range potential $V(\mathbf{r})$. In particular, show that the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int V(\mathbf{r})\,\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r}\right|^2$$

where $\hbar\mathbf{q} = \hbar(\mathbf{k}_i - \mathbf{k}_f)$ is the momentum transferred, $\hbar\mathbf{k}_i$ and $\hbar\mathbf{k}_f$ being the initial and final momenta respectively. **[9]**

Show for the Yukawa potential $V(r) = V_0 \dfrac{\mathrm{e}^{-\mu r}}{r}$, $\mu > 0$, that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4m^2V_0^2}{\hbar^4\left(q^2+\mu^2\right)^2}.$$ **[6]**

Hence show that the total cross section, σ, satisfies

$$\sigma \overset{E\to\infty}{\sim} A\,E^{-1},$$ **[5]**

with A a constant.

Note: $\displaystyle\int_0^\infty \mathrm{e}^{-\alpha r}\sin\beta r\,\mathrm{d}r = \frac{\beta}{\alpha^2+\beta^2}$, for $\alpha > 0$.

4. Explain what is meant by the terms, *state vector, coordinate representation* and *wave function* as applied to a quantum system. **[3]**

The operators x and p representing the observables associated with the measurement of a particle's position x' and momentum p' in one-dimension have eigenvalue equations $x|x'\rangle = x'|x'\rangle$ and $p|p'\rangle = p'|p'\rangle$ and obey the commutation relation $[x,p] = i\hbar$. Show that the matrix elements of x in the momentum representation can be expressed in terms of the Dirac δ-function as

$$\langle p''|x|p'\rangle = i\hbar \frac{\partial}{\partial p''}\delta\left(p'' - p'\right).$$

Hence, by considering the expectation value of the operator x in the state $|\psi\rangle$ and using the closure relation $\int |p'\rangle \mathrm{d}p' \langle p'| = 1$, deduce that in the momentum representation

$$x = i\hbar \frac{\partial}{\partial p}\ .$$ **[4]**

Show that the transformation coefficients $\langle p'|x'\rangle$ for a change from the momentum basis to the coordinate basis are

$$\langle p'|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{-ip'x'/\hbar}.$$ **[5]**

The Schrödinger equation for the state vector $|\psi(t)\rangle$ of a spinless particle of momentum p and mass m moving in a potential $V(x)$ in one-dimension is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left[\frac{p^2}{2m} + V(x)\right]|\psi(t)\rangle.$$

By taking its projection onto the ket $|p\rangle$, i.e. forming $\phi(p,t) = \langle p|\psi(t)\rangle$, show that in the momentum representation, the Schrödinger equation takes the form of an integro-differential equation

$$i\hbar\frac{\partial\phi(p,t)}{\partial t} = \frac{p^2}{2m}\phi(p,t) + \int_{-\infty}^{\infty}\tilde{V}(p-p')\,\phi(p',t)\,\mathrm{d}p'\ ,$$

where

$$\tilde{V}(p-p') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{e}^{-i(p-p')x/\hbar}V(x)\,\mathrm{d}x.$$ **[4]**

Show, by considering a suitable change of variables, that the <u>time-independent</u> Schrödinger equation, $\left(\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2\right)|\psi\rangle = E|\psi\rangle$, for a linear harmonic oscillator has the same structure in momentum space as it has in coordinate space. Hence deduce that a normalised solution $\psi(x)$ of the wave equation in coordinate space can be related to a normalised solution of the wave equation in momentum space $\phi(p)$ by

$$\phi(p) = \frac{1}{\sqrt{m\omega}}\psi\left(\frac{p}{m\omega}\right).$$ **[4]**

Note the identities $x\delta'(x) = -\delta(x)$; $\int_{-\infty}^{\infty}\mathrm{e}^{i(p-p')x/\hbar}\mathrm{d}x = 2\pi\hbar\delta(p-p')$.

5. The wave function

$$\psi(r,\theta) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{ikr}}{r},$$

is a solution, for large r, of the Schrödinger equation describing the elastic scattering of a beam of spinless particles of mass m and momentum $\hbar\mathbf{k}$, by a short-range central potential $V(r)$. Identify the scattering amplitude and state without proof how it is related to the differential cross section $d\sigma/d\Omega$. **[1]**

The wave function $\psi(r,\theta)$ can be expanded in partial waves as

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} R_\ell(r)\, P_\ell(\cos\theta)$$

where $P_\ell(\cos\theta)$ are Legendre polynomials and $R_\ell(r)$ is the radial wave function satisfying the radial wave equation

$$\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r) + k^2\right\} R_\ell(r) = 0$$

with $k^2 = 2mE/\hbar^2$. Show that as $r \to \infty$,

$$R_\ell(r) \stackrel{r\to\infty}{\longrightarrow} \frac{d_\ell}{kr}\sin(kr - \ell\pi/2 + \delta_\ell)\ ,$$

where δ_ℓ is the phase shift for the ℓ-th partial wave and d_ℓ is a constant. **[4]**

You may assume the result that if $V(r) = 0$, then the linearly independent solutions of the radial equation are the regular spherical Bessel functions $j_\ell(kr)$ and the irregular spherical Neumann functions $n_\ell(kr)$ which have the asymptotic forms

$$j_\ell(kr) \stackrel{r\to\infty}{\longrightarrow} \frac{1}{kr}\sin(kr - \ell\pi/2) \qquad n_\ell(kr) \stackrel{r\to\infty}{\longrightarrow} -\frac{1}{kr}\cos(kr - \ell\pi/2)\,.$$

Express $\psi(r,\theta)$ as the sum of incoming and outgoing spherical waves. **[3]**

If $f(\theta)$ is also expanded as

$$f(\theta) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(\cos\theta)\ ,$$

and noting that the asymptotic form of Bauer's relation is

$$e^{i\mathbf{k}\cdot\mathbf{r}} \stackrel{r\to\infty}{\longrightarrow} \frac{1}{kr}\sum_{\ell=0}^{\infty}(2\ell+1)\, i^\ell \sin(kr - \ell\pi/2)\, P_\ell(\cos\theta)\,,$$

show, by considering the coefficients of the incoming and outgoing spherical waves in each partial wave, that

$$\begin{aligned} d_\ell &= (2\ell+1)\, i^\ell e^{i\delta_\ell}, \\ f_\ell &= (2\ell+1)\frac{e^{i\delta_\ell}\sin\delta_\ell}{k}. \end{aligned}$$ **[4]**

 QUESTION CONTINUES OVER PAGE

The beam of particles scatters from a repulsive potential

$$V(r) = \frac{A}{r^2} \qquad ; \qquad A > 0.$$

Solve the radial equation for an 'effective' centripetal barrier with λ defined by $\lambda(\lambda+1) = \ell(\ell+1) + 2mA/\hbar^2$. By comparing the asymptotic form of this solution with the general form, $\frac{1}{kr}\sin(kr - \ell\pi/2 + \delta_\ell)$, show that the phase shift, δ_ℓ, is given by

$$\delta_\ell = -\frac{\pi}{4}\left\{\left[(2\ell+1)^2 + \frac{8mA^2}{\hbar^2}\right]^{1/2} - (2\ell+1)\right\}. \qquad [5]$$

Hence show that, if $8mA^2/\hbar^2 \ll 1$, then the differential cross section

$$\frac{d\sigma}{d\Omega} \simeq \frac{\pi^2 m^2 A^2}{4\hbar^4 k^2}\frac{1}{\sin^2(\theta/2)}. \qquad [3]$$

Note: $\sum_{\ell=0}^{\infty} P_\ell(\cos\theta) = \frac{1}{2\sin(\theta/2)}$.

END OF PAPER